N/S
NASA
7N-90-TM
277404
P. 113

Institute for Space Studies

GAUGE COVARIANT THEORY OF GRAVITATION

V. Canuto

P. J. Adams

S.-H. Hsieh

E. Tsiang

(NASA-TM-103088) GAUGE COVARIANT THEORY OF GRAVITATION (NASA) 113 p

N90-70851

Unclas

00/90 0277404

GODDARD SPACE FLIGHT CENTER

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

GAUGE COVARIANT THEORY OF GRAVITATION

V. Canuto*, P. J. Adams**, S.-H. Hsieh, E. Tsiang†

Institute for Space Studies, NASA
New York, N. Y. 10025

*) Also with the Physics Dept., CCNY, New York

†) NAS-NRC Research Associate

**) Present Address: Physics Dept., University of British Columbia, Vancouver, V6T1W5

TABLE OF CONTENTS

Abstract

That the equations describing physical phenomena should be independent of the system of coordinates employed is a universally accepted principle. That the same equations should also be independent of the system of units employed (gauge invariance in Weyl terminology) has also been discussed but to a lesser extent due to the lack of experimental evidence. The situation has changed recently with the discovery that high energy phenomena exhibit scale invariance, i.e. invariance with respect to the chosen unit of length. Since the behavior of matter at high energy is relevant to our understanding of the early phases of cosmological evolution, we have studied the extension of scale invariance to gravitational phenomena. Gravitational equations will be derived that are valid in any system of units. The freedom of choosing an arbitrary system of measuring units is represented by the presence of an arbitrary gauge function $\beta(t)$. Standard Einstein equations are recovered if a particular gauge is imposed, $\beta(t)$ = const. Following Dirac, the corresponding units will be called Einstein or mechanical units.

Atomic units, derivable from atomic constants, are assumed to be distinct from Einstein units and consequently a different gauge condition must be imposed. It is suggested that Dirac's large number hypothesis be used for the determination of this condition so that gravitational phenomena can be described in atomic units. The result allows a natural interpretation of the possible variation of the gravitational constant without compromising the validity of general relativity.

A geometrical interpretation of the gauge covariant theory is possible if the covariant tensors in Riemannian space are replaced by covariant co-tensors in a

Integrable Weyl space. A gauge invariant action principle is constructed from the metrical potentials of the integrable Weyl space.

Application of the dynamical equations in atomic units to cosmology yields a family of homogeneous solutions characterized by $R \sim t$ for large cosmological times. This in turn implies $q_0 \cong 0$, in agreement with the most recent determination of the deceleration parameter.

Equations of motion in atomic units are solved for spherically symmetric gravitational fields. Expressions for perihelion shift and light deflection are derived. They do not differ from the predictions of general relativity except for secular variations, having the age of the Universe as a time scale. Similar variations of periods and radii for planetary orbits are also derived.

The generalized hydrodynamic equations derived for atomic units are studied. It is found that the stellar structure equations are formally unchanged, except that G and M can now be functions of the cosmological time. This in turn would imply secular variations of the stellar luminosities.

The effects of these results on the past climatology of the earth and other geological effects are discussed.

None of the consequences of the theory investigated so far is found to be in disagreement with observations.

I. Introduction

In his theories of relativity, Einstein recognized the equivalence of different states of motion and proceeded to build up physical theories which incorporated such equivalence by postulating a priori symmetry principles. The result is a beautiful, far reaching new structure for classical physics. Various predictions of the theories, seemingly parodixical at the time, have been successfully verified and consequently man's conception of space-time has been radically altered. Since the dawn of relativity, symmetry principles have played a dominant role in theoretical physics, most prominently in the field of elementary particles where space-time and internal symmetries are postulated a priori. In classical physics however, the space-time symmetries considered by Einstein seemed exhaustive and there had been few attempts to impose further invariance conditions.

In an attempt to unify electromagnetism with gravitation, Weyl(1) generalized Riemannian geometry by allowing lengths to change under parallel displacement. Although the theory was soon rejected as unphysical(2), a mathematical operation known as gauge transformation was introduced, which with some modification has been widely used in physics. As was pointed out by Eddington(3), a gauge transformation represents a change of units of measurement and hence gives a general scaling of the physical system being investigated. In recent years, due to the scaling behavior exhibited in high energy particle scattering experiments, there has been considerable interests in manifestly gauge invariant theories(4). However, such invariance is considered valid only in the limit of high energies or vanishing rest mass. This is due to the fact that in elementary particle theories rest masses must be considered constants, and it is well known(5) that gauge covariance is generally valid only when the constant rest mass requirement is relaxed. It is of course sensible to assume that particle rest masses are constants provided the units of measure are the elementary particles themselves. Under a general gauge transformation, the units of measure are altered and the constant rest mass condition no longer seems physically necessary. In this connection we point out that it may be felt that atomic physics has provided us a unique system of units whereby all physical quantities are measured, and it is therefore unphysical to consider general transformations away from the atomic units. We do not agree with this point of view. In classical experiments concerning gravitational interaction, masses and lengths of arbitrary macroscopic objects have been used as units of measure. Whether such units are in fact constant multiples of atomic units, as conventionally assumed, has not been established. Indeed it was the recognition of the possibility of a temporal dependence of the proportionality factors between atomic units and gravitational units of bulk matter that led Dirac[6] to formulate his large number hypothesis (LNH), to which we shall address ourselves presently. In this paper we consider the behavior of Einstein's theory of gravitation under an arbitrary transformation of units. It will be seen that this leads naturally to a gauge covariant theory of gravitation originally proposed by Dirac[7]. We shall present and develop the theory and show how naturally Dirac's LNH can be fitted into the structure of this theory.

The recent revival of interest in Dirac's large number hypothesis and the ensuing cosmological models has been due to the re-introduction by Dirac[8] of the concept of two metrics and of the possibility of continuous matter creation as a modification of the earlier version of LNH[9]. These modifications allow plausible interpretation of observational data in terms of LNH while at the same time preserving the validity of Einstein's theory of gravitation which has survived improved experimental tests in the past few years. The present version of LNH and some of its consequences have been discussed and amplified in many of Dirac's recent publications[10] [11], the essence of which was summarized by Canuto and Lodenquai[12]. We have gathered the salient features of Dirac's LNH in Appendix II. Here we point out that the LNH yields an asymptotic theory, valid for large cosmological times. For example, in cosmology it predicts that the scale factor $R(t)$ in a spatially homogeneous universe should be proportional to t, to within a slowly varying function of t, such as $\ln t$[11], where t is the cosmic time in atomic units. This specifies the kinematics of the present universe upon which cosmological tests of the recent past can be made[12] [13]. However, it is incorrect to extend the present kinematics $R(t) \sim t$ back to the early phase. Indeed, when this is done, it is[14] found that the strong interaction rate was so slow that nucleosynthesis could not have taken place. Nor could the background radiation at any time be in equilibrium with matter because the photon mean free time for compton scattering is always longer than the evolution time scale of the universe. We emphasize that this in no way implies a contradiction between the LNH and observation. As mentioned above, Steigman's[14] result is not a necessary consequence of LNH.

On the other hand, in problems concerning local gravitational phenomena such as planetary orbits, Dirac[10] used various intuitive arguments to derive consequences of his LNH to contrast with those of the standard gravitational theory.
Others have improvised such methods[15] and in the process dynamical equations have been posited ad hoc. The results so derived again cannot be taken as consequences of the LNH.

While it is true that the LNH does not represent a complete physical theory and that alternative interpretations of the large numbers exist which seem to render the LNH unnecessary, we feel that no argument showing the LNH to be either contradictory to observational data or logically inconsistent has yet been put forward. To finally reject or embrace the LNH, a dynamic theory which incorporates the LNH should be developed and the consequences systematically investigated. This is the purpose of the present paper.

In II, we shall develop a gauge covariant theory of gravitation which, in addition to the general coordinate invariance imposed by Einstein, contains invariance under scale transformation. Thus, not only are physical laws invariant for observers with different states of motion, they are also invariant for observers with different measuring instruments. In practice two kinds of units have been used, the gravitational units we mentioned above, which in the sequel we call "Einstein Units" in conformity with Dirac's nomenclature, and atomic units. We shall see that within the gauge covariant theory of gravitation the LNH finds a natural role: it gives the relation between atomic and Einstein units. In our opinion this was precisely the intent of Dirac's LNH for the past 40 years.

A gauge invariant variational principle will be given from which field equations will be derived. However, to facilitate the derivation of conservation equations we find it convenient to borrow the mathematics developed for Weyl[1] space and to use the notion of co-covariant equations in Weyl space as introduced by Dirac.[7] The elementary features of Weyl space and some relevent mathematical formulae are collected in Appendix I. It should be noted that we do not use the geometry of Weyl space for the purpose of unifying the electro-magnetic and gravitational fields as Dirac[7] intended.

In fact, in the present paper, we shall not consider the existence of a coherent electromagnetic field at all. Incoherent fields can be considered a photon gas and can be included in the matter part of the energy momentum tensor. While our variational principle is identical to the gravitational part of Dirac's[7] action, our treatment allows us to be more specific about the gauge function and its connection with LNH. Furthermore, we can also include matter creation which is absent in Dirac's formulation of the action principle.

Due to the apparent similarity of Dirac's action integral with that of Brans and Dicke[22] and its variants[23], it could be erroneously assumed that Dirac's theory is either contained in or is another variant of the Brans-Dicke theory. We have thus devoted one section to explain the difference in both spirit and content of the theories.

Having obtained a set of dynamical equations we can derive astrophysical consequences in analogy with the standard theory, viz. dynamical equations are solved for specific problems at hand. In particular, we consider homogeneous cosmological solutions in III. In IV, we study the geodesic equations and derive expressions for the perihelion shifts, light deflections and secular variations of planetary orbital elements. We also derive the stellar structure equation for a star in quasi-static equilibrium. A short discussion of the effects of the above results on the past thermal history of the earth as well as of other geophysical effects is given at the end.

II. Gauge Covariant Theory of Gravitation

Einstein's general theory of relativity assumes that the gravitational constant G is a true constant. The LNH states that $G \sim t^{-1}$, where t is the cosmological epoch. To reconcile these Dirac[(6) (8)] introduced the concept of two metrics as follows. The Einstein equations with constant G are assumed to be valid in "mechanical", or Einstein units. They are necessarily different from atomic units upon which the LNH was constructed. It is only in atomic units that G has an epochal dependence. Clearly, the scale factor which effects a transformation between Einstein and atomic units must be time dependent. To obtain relevant dynamical equations in atomic units, Dirac suggests a transformation of the line element, $ds_E = \beta \, ds_A$, be made and the corresponding field equations derived as a result. However, the functional form of β is not specified. We shall see below that it can be limited by the present prediction of LNH.

When units transformations of the kind mentioned above are considered, a natural generalization of Einstein's theory of gravitation is obtained by requiring that physical laws be invariant under general units transformation as well as coordinate transformation. An action principle satisfying these invariance requirements will be given. Gauge covariant field equations can then be derived. We shall motivate the construction of such a theory by the following analysis of Einstein equations under a general transformation of units.

2.1 Transformation of Units - Einstein Field Equations

We start with the Einstein equations in Einstein units

$$\bar{G}_{\mu\nu} = -8\pi \bar{\mathfrak{T}}_{\mu\nu} + \bar{\Lambda}\, \bar{g}_{\mu\nu} \tag{2.1}$$

where

$$\bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \tfrac{1}{2}\, \bar{g}_{\mu\nu}\, \bar{R} \tag{2.2}$$

is the Einstein tensor. The bars indicate that Einstein units are being used. The line element $d\bar{s}$ is given by

$$d\bar{s} = \bar{g}_{\mu\nu}\, dx^{\mu}\, dx^{\nu} \tag{2.3}$$

where the coordinate interval is dimensionless. $\bar{\mathfrak{T}}_{\mu\nu}$ is the matter energy-momentum tensor expressed in geometric units, i.e. $(\text{length})^{-2}$, with lengths in Einstein units. Under a transformation

$$d\bar{s} \rightarrow ds = \beta^{-1}(x)\, d\bar{s} \tag{2.4}$$

it is easily seen that since

$$ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} \tag{2.5}$$

then

$$\bar{g}_{\mu\nu} = \beta^2 g_{\mu\nu} \tag{2.6}$$

Equation (2.6) represents a conformal transformation from a geometry described by $\bar{g}_{\mu\nu}$ to one described by $g_{\mu\nu}$. The corresponding transformation of the Ricci tensors and therefore of the Einstein tensors is well known(16). We have*)

$$\bar{G}_{\mu\nu} = G_{\mu\nu} + \frac{2\beta_{\mu;\nu}}{\beta} - \frac{4\beta_{\mu}\,\beta_{\nu}}{\beta^2} - g_{\mu\nu}\left(2\,\frac{\beta^{\lambda}_{;\lambda}}{\beta} - \frac{\beta^{\lambda}\,\beta_{\lambda}}{\beta^2}\right) \tag{2.7}$$

where on the RHS, covariant differentiation as well as index raising and lowering operations are carried out with respect to $g_{\mu\nu}$.

The cosmological term can be written as

$$\bar{\Lambda}\,\bar{g}_{\mu\nu} = \bar{\Lambda}\,\beta^2\,g_{\mu\nu} = \Lambda\,g_{\mu\nu} \tag{2.8}$$

with

$$\Lambda = \beta^2\,\bar{\Lambda} \tag{2.9}$$

*)For any scalar α, $\alpha_{\mu} \equiv \alpha_{;\mu}$.

This does not complete the transformation of units on the Einstein equations since the consideration of conformal transformation of geometries does not tell us how $\bar{\mathfrak{J}}_{\mu\nu}$ transforms. To find out, we consider a further transformation of (2.7). Let

$$g_{\mu\nu} = \varphi^2 g'_{\mu\nu} \tag{2.6a}$$

Denoting covariant differentiation w.r.t. $g'_{\mu\nu}$ by ":", the expression on the RHS of (2.7) can be written as

$$G'_{\mu\nu} + 2\frac{\varphi_{\mu:\nu}}{\varphi} - 4\frac{\varphi_\mu \varphi_\nu}{\varphi^2} + g'_{\mu\nu}\left(2\frac{\varphi^\lambda_{:\lambda}}{\varphi} - \frac{\varphi^\lambda \varphi_\lambda}{\varphi^2}\right)$$

$$+ 2\frac{\beta_{\mu:\nu}}{\beta} - 4\frac{\beta_\mu \varphi_\nu}{\beta\varphi} + 2g'_{\mu\nu}\frac{\varphi^\lambda \beta_\lambda}{\beta\varphi}$$

$$- 4\frac{\beta_\mu \beta_\nu}{\beta^2} - g'_{\mu\nu}\left(2\frac{\beta^\lambda_{:\lambda}}{\beta} + 4\frac{\beta^\lambda \varphi_\lambda}{\beta\varphi} - \frac{\beta^\lambda \beta_\lambda}{\beta^2}\right)$$

$$= G'_{\mu\nu} + 2\frac{(\beta\varphi)_{\mu:\nu}}{\beta\varphi} - 4\frac{(\beta\varphi)_\mu(\beta\varphi)_\nu}{(\beta\varphi)^2} - g'_{\mu\nu}\left(2\frac{(\beta\varphi)^\lambda_{:\lambda}}{\beta\varphi} - \frac{(\beta\varphi)^\lambda(\beta\varphi)_\lambda}{(\beta\varphi)^2}\right)$$

Noting that

$$\bar{g}_{\mu\nu} = \beta^2 g_{\mu\nu} = \beta^2 \varphi^2 g'_{\mu\nu} = \beta'^2 g'_{\mu\nu}$$

we see that the above exercise demonstrates the form invariance of the RHS of (2.7). Similar invariance can be trivially ascertained for the cosmological term in the Einstein equation. We conclude therefore that the matter source term must also be form invariant and that the field equations in general units can be written as

$$G_{\mu\nu} + 2\,\frac{\beta_{\mu;\nu}}{\beta} - 4\,\frac{\beta_\mu\,\beta_\nu}{\beta^2} - g_{\mu\nu}\left(2\,\frac{\beta^\lambda_{;\lambda}}{\beta} - \frac{\beta^\lambda\,\beta_\lambda}{\beta^2}\right) = -\,8\pi\,\mathfrak{J}_{\mu\nu} + \Lambda\,g_{\mu\nu} \tag{2.10}$$

As noted above, $\bar{\mathfrak{J}}_{\mu\nu}$ and $\mathfrak{J}_{\mu\nu}$ are expressed in geometric units and the gravitational constant does not appear explicitly. If conventional units are used for the stress-energy tensor, we can write

$$\bar{\mathfrak{J}}_{\mu\nu} = \bar{G}\,\bar{T}_{\mu\nu} \tag{2.11a}$$

$$\mathfrak{J}_{\mu\nu} = G\,T_{\mu\nu} \tag{2.11b}$$

While $\bar{G}$ has been stipulated to be constant, G in general is not, unless the scale factor β is constant. In particular, if $T_{\mu\nu}$ is measured in atomic units, one gets from (2.10) a numerical value in atomic units for G which can be expected to vary. It should be remarked that in physical measurements involving the gravitational interaction alone, the gravitational constant and the source strength (mass) always appear as a product and cannot be measured separately. Conventionally, one defines a macroscopic unit mass. G can then be measured in terms of it.

2.2 Co-Covariant Equations - Geodesic Equations

Having arrived at the gravitational field equations (2.10) in general units we seek to characterize the nature of the spacetime manifold underlying such equations. In general relativity spacetime is taken to be Riemannian and any needed equation can be found by taking the pertinent equation from special relativity and writing it so that it is form invariant under arbitrary coordinate transformations. We will see that the spacetime underlying eq. (2.10) is an integrable Weyl (IW) manifold, and that any needed equation can be found by taking the pertinent equation in special relativity and writing it so that it is form invariant under both arbitrary coordinate and arbitrary scale transformations. Whereas equations form invariant under arbitrary coordinate transformations are called covariant, equations form invariant under both arbitrary coordinate and arbitrary scale transformations will be called co-covariant after Dirac[7]. Use of the term covariant is reserved for properties related exclusively to the metric tensor $g_{\mu\nu}$ as in Riemannian theory.

In Riemannian geometry, if a displacement vector δx^{μ} is parallel transported, its length does not change along the path. Thus

$$d(g_{\mu\nu}\,\delta x^{\mu}\,\delta x^{\nu}) = 0 \tag{2.12}$$

However, under a general scale transformation,

$$ds \rightarrow ds' = \varphi\, ds \tag{2.13}$$

the metric tensor becomes

$$g'_{\mu\nu} = \varphi^2 g_{\mu\nu} \tag{2.14}$$

The length of the displacement vector in this new system of units will generally change under parallel transport. In fact,

$$d(g'_{\mu\nu} \delta x^\mu \delta x^\nu) = 2 g'_{\mu\nu} \delta x^\mu \delta x^\nu d(\ln \varphi) \tag{2.15}$$

Consequently, a generalization of Riemannian geometry is called for. Such a generalization was provided by Weyl[1] and we shall use the mathematics developed for this generalized geometry to describe our gauge covariant theory of gravitation. We have given a concise summary of the essential features of Weyl's geometry in Appendix I. More details can be found in the books by Eddington[3] or Weyl[1], himself.

It should be pointed out that Einstein[17] had objected to the use of Weyl geometry to describe the physics of electro-magnetic as well as gravitational phenomena. The essence of the objection[18] rests on the fact that sharp spectral lines are observed even in the presence of electro-magnetic field, whereas in Weyl's theory, the electro-magnetic field would imply a non-integrable length which in turn implies that different atoms, having very different past world lines, should not be emitting radiation at the same frequency. The same objection still applies even though a different system of units can be set up, since transformation of units such as given by (2.13) does not alter the gauge invariant integrability condition (see Appendix II)

$$k_{\mu;\nu} - k_{\nu;\mu} = 0 \qquad (2.16)$$

However, in order to include gauge covariance considerations of gravitational phenomena, we do not need the fully generalized Weyl space. Indeed, comparing (2.15) with (A1.3), we need generalize the Riemannian geometry to the extent that Weyl's metric vector k_μ can be expressed as a gradient

$$k_\mu = \Phi_{,\mu} \qquad (2.17)$$

in which case (2.16) is satisfied and Einstein's objection does not affect our use of such an Integrable Weyl geometry (IW geometry). In the literature, one often finds statements to the effect that whenever (2.16) is satisfied, the geometry is Riemannian. It is true that when (2.17) holds, the space is conformally equivalent to a Riemannian space. However, to identify the two is to assert that k_μ is unobservable and is completely irrelevant to the description of the physical world. We do believe that an "absolute" k_μ has no physical significance and hence is unobservable. In fact, this is the reason for imposing gauge invariance[19]. But the relative k_μ which describes the difference between two systems of units, such as those provided by gravitational theory and atomic theory does have physical significance. The non-measurability of the "absolute" metric vector allows one to stipulate that k_μ is identically zero in one system of units which we choose to be Einstein units.

Using general units of measure, the natural description of gravitational phenomena is given by the IW space whose metrical properties are given by the metric tensor

$g_{\mu\nu}$ and a scalar potential Φ. But to make use of the mathematics developed for Weyl geometry, it is convenient to retain the scale vector k_μ with the understanding that it is a gradient vector field.

Having determined the mathematical space for our description of physical phenomena, we recall how symmetry principles have been powerful tools for arriving at general equations in the theory of relativity when equations in special cases are known. Lorentz covariance requires that all physical equations are written as tensor equations in Minkowski space. General covariance then requires that the tensor equations must involve tensors in Riemannian space. Thus, having postulated the symmetry properties, equations of motion written in a particular coordinate system can be generalized by putting the quantities involved in the appropriate tensor form so that the equations are valid in arbitrary coordinates. In particular, we note that it has been a common practice to generalize special relativistic equations to be generally covariant by replacing partial derivatives by covariant derivatives. We shall apply the same technique to further generalize Einstein's theory of gravitation. From the preceding discussion, we expect that in a gauge covariant theory, the physical equations must involve tensors in Weyl space, called co-tensors.

The notion of a co-tensor and its power, and the concept of co-covariant differentiation which brings a co-tensor into a co-tensor of the same power are described in Appendix I. We observe here that $\beta(x)$, defined in (2.4) as the scale factor between Einstein units and any general units, is a co-scalar of power -1. That this is indeed the case can be seen trivially by considering transformation of the type (2.13)

$$ds' = \beta'^{-1}\, d\bar{s} = \varphi\, ds = \varphi\, \beta^{-1}\, d\bar{s} \tag{2.18}$$

Hence

$$\beta' = \varphi^{-1} \beta . \tag{2.19}$$

From considerations of the transformation properties of the Einstein equation in the preceding section, we expect that the field equation in a gauge covariant theory can be written as an in-tensor equation having the form

$$*G_{\mu\nu} = - 8\pi *\mathfrak{T}_{\mu\nu} + *\Lambda_{\mu\nu} \tag{2.20}$$

where $*\Lambda_{\mu\nu}$ is the cosmological term and each term is an in-tensor. We consider them separately. Since (2.20) is the generalization of (2.1) the in-tensors must be the generalizations of the tensors in (2.1). Since the metric tensor is a co-tensor of power +2, we can write

$$*\Lambda_{\mu\nu} = \Lambda g_{\mu\nu} \tag{2.21}$$

with

$$\Lambda = \beta^2 \bar{\Lambda} \tag{2.22}$$

being a co-scalar of power -2. The Einstein tensor consists of two terms, each having its in-invariant generalization. From (A1.7) and (A1.8) we can write

$$
\begin{aligned}
{}^{*}G_{\mu\nu} &= {}^{*}R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}\, {}^{*}R \\
&= G_{\mu\nu} - k_{\mu;\nu} - k_{\nu;\mu} + 2\, g_{\mu\nu}\, k^{\lambda}{}_{;\lambda} - g_{\mu\nu}\, k^{\lambda}\, k_{\lambda} - 2\, k_{\mu}\, k_{\nu}
\end{aligned}
\tag{2.23}
$$

where the term $(k_{\mu;\nu} - k_{\nu;\mu})$ has been dropped because (2.17) is assumed. For the energy-momentum tensor we consider for simplicity a pressureless fluid. In general relativity we have

$$\bar{\mathfrak{T}}_{\mu\nu} = \bar{G}\, \bar{\rho}\, \bar{u}_{\mu}\, \bar{u}_{\nu}$$

The four-velocity can be easily shown to be a co-vector of power +1. The in-tensor generalization of the above can then be written as

$${}^{*}\mathfrak{T}_{\mu\nu} = (G\rho)\, u_{\mu}\, \mu_{\nu} \tag{2.24}$$

where $(G\rho)$ is a co-scalar of power -2. The separation of G and ρ is artificial at this stage for we do not know a priori how G and ρ transforms under a scale transformation. For the present, we use (2.24) formally and (2.20) can be written as

$$G_{\mu\nu} - k_{\mu;\nu} - k_{\nu;\mu} + 2\, g_{\mu\nu}\, k^{\lambda}_{\ ;\lambda} - g_{\mu\nu}\, k^{\lambda}\, k_{\lambda} - 2\, k_{\mu}\, k_{\nu}$$

$$= -\, 8\pi\, G\rho\, u_{\mu}\, \mu_{\nu} + \Lambda\, g_{\mu\nu} \qquad (2.20a)$$

Equation (2.20a) is identical to (2.10) if

$$k_{\mu} = -\, \partial_{\mu} (\ell n\ \beta) = -\, \frac{\beta_{,\mu}}{\beta} \qquad (2.25)$$

This amounts to prescribing the gauge potential of IW space as follows: In Einstein units, the natural gauge

$$k_{\mu} = 0\, ;\ \Phi = \text{constant}$$

is used. For any other system of units, the gauge must be changed, and the gauge induced by such a change of units is precisely (2.25). Thus, in general the metric potential Φ must be written as

$$\Phi = -\, \ell n\ \beta \qquad (2.26)$$

where β is the scale factor between the units being used and the Einstein units.

In addition to the field equation (2.20), one can easily generalize other equations in relativity for the gauge covariant theory. Thus, the conservation equation

$$\mathfrak{J}^{\mu\nu}{}_{;\nu} = 0$$

which follows from the Einstein equations, must now be written as

$$*\mathfrak{J}^{\mu\nu}{}_{*\nu} = 0 \tag{2.27}$$

where $*\mathfrak{J}^{\mu\nu}$ is a co-tensor of power -4.

We shall now derive the geodesic equations for massive particles as well as for photons.

We shall follow here the method of Papapetrou[20]. Given the general expression (A1.1) for the parallel transport defined by the connection $\overset{*}{\Gamma}{}^{\mu}_{\alpha\beta}$, the general equation can be easily derived

$$\frac{d^2 x^\mu}{d\lambda^2} + \overset{*}{\Gamma}{}^{\mu}_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = f(\lambda) \frac{dx^\mu}{d\lambda}$$

where λ is a parameter characterizing the curve. The scalar $f(\lambda)$ can be easily determined by multiplying the previous equation by

$$g_{\mu\lambda}\, \xi^\lambda \;, \quad \xi^\mu = \frac{dx^\mu}{d\lambda}$$

and calling

$$g_{\mu\lambda}\, \xi^{\lambda}\, \xi^{\mu} = \epsilon = \text{const.}$$

The result is

$$f(\lambda) = -\, k_{\mu}\, \xi^{\mu}$$

which allows us to write geodetic equation in the general form

$$\frac{d\xi^{\mu}}{d\lambda} + \Gamma^{\mu}_{\alpha\beta}\, \xi^{\alpha}\, \xi^{\beta} = -\, k_{\nu}\, (\epsilon\, g^{\mu\nu} - \xi^{\mu}\, \xi^{\nu})$$

For massive particles we can identify $d\lambda = ds$

$$\xi^{\lambda} = \frac{dx^{\mu}}{d\lambda} \rightarrow \frac{dx^{\mu}}{ds} = u^{\mu}$$

and moreover $\epsilon = 1$.

The final form of the geodetic equation is

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\alpha\beta}\, u^\alpha u^\beta = \frac{\beta_{,\nu}}{\beta}\,(g^{\mu\nu} - u^\mu u^\nu) \qquad (2.28a)$$

because of (2.25).

For photons, since $ds \to o$, ξ^λ cannot be identified with a velocity and we shall only impose

$$\epsilon = o$$

so that the final equation reads

$$\frac{d\xi^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}\, \xi^\alpha \xi^\beta = - \frac{\beta_{,\nu}}{\beta}\, \xi^\mu \xi^\nu \qquad (2.28b)$$

In the case of massive particles Eq. (2.28a) can also be derived by generalizing the ordinary geodesic equation

$$\bar{u}^\mu_{;\nu}\, \bar{u}^\nu = o$$

to its co-covariant form

$$u^\mu_{*\nu}\, u^\nu = o$$

Using the definition (A1.16) and remembering that u^μ is a covector of power -1, it is easy to recover (2.28a).

In summary, to obtain the generalized equations in the gauge covariant theory, we take the general relativistic equations, write the tensors in co-tensor form, and replace covariant differentiations by co-covariant differentiations. It should be noted that in addition to the variables that exist in the general relativistic equation, we now have also β, whose functional form is not specified. We shall return to this subject after a gauge invariant variational principle, from which (2.20) can be derived, has been introduced. The physical interpretation of equations (2.27) and (2.28) as well as a possible way to determine β for the above equations will be discussed after the formal development of the theory is completed.

2.3 Gauge Invariant Action Principle

By analogy with the general relativistic case, the action must be an in-invariant constructed out of the metric tensor $g_{\mu\nu}$ and the metric potential ϕ. Since we have imposed the natural gauge for Einstein units, the potential can be replaced by β. The simplest way to proceed is to generalize Einstein's action in the same manner we constructed co-covariant equations in the previous section. Hence consider the quantities involved in

$$\bar{I} = \int \bar{R} \sqrt{\bar{g}} \; d^4 x \tag{2.29}$$

where $\quad \bar{g} \equiv |\det(\bar{g}_{\mu\nu})|$

The co-scalar generalization of $\bar{R}$ is $*R$ which has power -2. Since $\sqrt{\bar{g}}$ is a co-scalar density of power $+4$, the natural generalization of (2.29) is

$$\int \beta^2 \, *R \sqrt{g} \; d^4 x \tag{2.29a}$$

where

$$g \equiv |\det(g_{\mu\nu})|$$

Conforming to the usual practice, we do not specify the matter Lagrangian $\mathfrak{L}$ aside from stipulating that it is a co-scalar with $\Pi = -4$ and that

$$
{}^{*}\mathfrak{J}^{\mu\nu} = g^{\mu\lambda} g^{\nu\rho} \, {}^{*}\mathfrak{J}_{\lambda\rho} = \beta^{-2} \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}} (\sqrt{g} \; \mathfrak{L}) \tag{2.32}
$$

(2.32) is the natural generalization of

$$
\bar{\mathfrak{J}}^{\mu\nu} = \frac{2}{\sqrt{\bar{g}}} \frac{\delta}{\delta \bar{g}_{\mu\nu}} \left(\sqrt{\bar{g}} \; \mathfrak{L} \right)
$$

in general relativity. The factor β^{-2} is necessary for the co-tensor power of both sides of (2.32) to be -4. We note that the first line of (2.30) is manifestly gauge invariant. The second line has been written out because it is easier to derive the field equations from and it will serve as a basis for our comparison with other theories.

Independently varying $g_{\mu\nu}$ and β, and using (2.30)-(2.32) we find

$$
\begin{aligned}
\delta I = \int d^4 x \sqrt{g} \Big\{ & \beta^2 \Big[R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R + 2 \frac{\beta^{\mu;\nu}}{\beta} - 4 \frac{\beta^{\mu} \beta^{\nu}}{\beta^2} \\
& - g^{\mu\nu} \left(2 \frac{\beta^{\lambda}_{;\lambda}}{\beta} - \frac{\beta^{\lambda} \beta_{\lambda}}{\beta^2} \right) + 8\pi \, {}^{*}\mathfrak{J}^{\mu\nu} + \frac{c}{2} \beta^2 g^{\mu\nu} \Big] \delta g_{\mu\nu} \\
& + \Big[4c \beta^3 + 16\pi \frac{\delta \mathfrak{L}}{\delta \beta} - 2 \beta R - 12 \beta^{\mu}_{;\mu} \Big] \delta \beta \Big\}
\end{aligned} \tag{2.33}
$$

The in-invariant character of (2.29a) is clear since, as it has been shown before, β is a co-scalar of power -1. In principle, one can add to (2.29a) terms involving co-covariant derivitives of β, and a term quartic in β, so that (c, c_1 = constants)

$$I' = \int dx^4 \sqrt{g} \; \{- \beta^2 \, {}^*R + c_1 \beta^{*\mu} \beta_{*\mu} + c \, \beta^4\} \tag{2.29b}$$

The in-invariance requirement dictates that only the quartic term can appear. The middle term while having the correct invariance properties has no contribution in our theory because

$$\beta_{*\mu} = \beta_{,\mu} - \Pi \, k_\mu \, \beta = 0$$

where Π is the power of β.

The first equality follows from the definition (A1.14a). The second equality follows from (2.25) and the fact that $\Pi = -1$ for β. Including a matter Lagrangian, we can then state our action principle as follows:

$$\begin{aligned} I &= \int d^4x \sqrt{g} \; \{- \beta^2 \, {}^*R + c \, \beta^4 + 16\pi \, \mathcal{L}\} \\ &= \int d^4x \sqrt{g} \; \{- \beta^2 R + 6 \, \beta^\mu \beta_\mu + c \, \beta^4 + 16\pi \, \mathcal{L}\} \end{aligned} \tag{2.30}$$

and $\delta I = 0$ (2.31)

Hence,

$$R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R + 2 \frac{\beta^{\mu;\nu}}{\beta} - 4 \frac{\beta^{\mu}\beta^{\nu}}{\beta^2} - g^{\mu\nu}\left(2 \frac{\beta^{\lambda}_{;\lambda}}{\beta} - \frac{\beta^{\lambda}\beta_{\lambda}}{\beta^2}\right)$$

$$= - 8\pi \, {}^{*}\mathfrak{J}^{\mu\nu} + \Lambda g^{\mu\nu} \tag{2.34}$$

$$\beta R + 6 \beta^{\mu}_{;\mu} = - 4 \Lambda \beta + 8\pi \frac{\delta \mathfrak{L}}{\delta \beta} \tag{2.35}$$

where we have put $\frac{c}{2} \beta^2 = - \Lambda$. Eq. (2.34) is seen to be identical to the Einstein equation we have derived previously for general units. Although (2.35) appears to be an independent field equation, we shall show that this is not the case. In vacuum,

$$\mathfrak{L} = 0 \;\; ; \;\; {}^{*}\mathfrak{J}^{\mu\nu} = 0 \;,$$

and it can be easily seen that the trace of (2.34) is identical to (2.35). More generally, the trace of (2.34) can be written as

$$\beta R + 6 \beta^{\mu}_{;\mu} = - 4 \Lambda \beta + 8\pi \beta \, {}^{*}\mathfrak{J}^{\mu}_{\mu} \tag{2.34a}$$

Comparison with (2.35) gives

$$\beta \, {}^{*}\mathfrak{J}^{\mu}_{\mu} = \frac{\delta \mathfrak{L}}{\delta \beta} \tag{2.36}$$

But this relation must be an identity by construction if I is to be gauge invariant. To see this, we consider an infinitesimal gauge transformation.

$$ds \rightarrow ds' = (1 + \lambda)ds$$

so that

$$\delta g_{\mu\nu} = 2\lambda\, g_{\mu\nu}$$

$$\delta \beta = - \lambda \beta$$

When the above variations are put into (2.33) and δI is required to vanish under such transformations, we find exactly (2.36). We remark that if Dirac's[7] matter Lagrangian

$$\mathfrak{L} = \beta \rho \quad ,$$

is taken one finds

$$*\mathfrak{J}^{\mu\nu} = \beta^{-1} \rho u^{\mu} u^{\nu}$$

and (2.36) is satisfied in this special case.

We have, in the above, derived the gravitational field equations in general units in three ways: a) by considering the effects of general units transformation on the Einstein equations, b) by co-covariant generalization of the Einstein equations and c) from a variational principle. The fundamental assumptions in all the above methods are of course the same. We have imposed gauge covariance as well as general coordinate covariance. In the first two methods the indeterminancy of β is clear, since it was introduced as an arbitrary scale factor and no new equation could be derived for its determination. With the variational method, although a new equation was obtained, it has been shown not to be independent of the rest of the field equations. Even though an underdetermined system of equations may seem an undesirable feature in the theory, we must recall again in this connection the role of general covariance in Einstein's theory. The field equation (2.1) has only six independent components, due to the Bianchi identities. This leaves four of the ten components of $g_{\mu\nu}$ undetermined, and one can choose a specific coordinate system by imposing coordinate conditions. By the same token, one can choose to use any arbitrary units by imposing a gauge condition. Just as the appropriate coordinate system is determined by the motion of the observer, the appropriate gauge is determined by the observing instruments. They can never be self-consistently prescribed within a covariant theory. Now we can understand how Dirac's LNH can be fitted into the structure of a gauge covariant theory of gravitation. It specifies the Einstein gauge which yields the mechanical units for which the general relativistic equations are valid. More importantly, it uses the numerical coincidences which have been derived by comparing Einstein and atomic units to determine the atomic gauge relative to the Einstein gauge. The LNH, which has no place in general relativity, becomes the observational input for specifying the gauge condition. Such input is necessary in the same sense that observation is necessary to identify laboratory coordinates in general relativity. We shall describe a determination of β in this manner after the conservation equations have been presented.

It should be remarked that when Dirac[7] introduced his action principles in the form of (2.29b), β was considered a new scalar field in addition to Weyl's metric vector k_μ. Furthermore, since Dirac's k_μ is not related to β by (2.25), $\beta_{*\mu}\ \beta^{*\mu}$ does not vanish. With this additional term in the Lagrangian, Dirac could ensure that no independent equation is derived for the variation of β, only if he puts $c_1 = 6$ in (2.29b). With our introduction of IW space and β as a metric potential, the indeterminacy of β becomes a natural consequence of the theory rather than a contrived situation. A variational principle formally identical to the one given by the second line of (2.30) was also considered by Bicknell[21].

2.4 Comparison with Scalar-Tensor Theories

The appearance of a scalar function β as well as $g_{\mu\nu}$ in our action integral (2.30) can give the erroneous impression that we are dealing with a special case of scalar-tensor theories. In this section, we shall point out the essential differences by comparing in detail our gauge covariant theory with the scalar-tensor theories of gravitation. For the latter, we shall consider in particular the theory of Brans and Dicke[22] (BD) and its generalization as given by Wagoner[23], which we shall review briefly.

The BD theory is characterized by the action[24]

$$I_{BD} = \int d^4x \sqrt{g} \left\{ -\varphi R - \omega \frac{\varphi_{,\mu} \varphi^{\mu}_{,}}{\varphi} + 16\pi \mathcal{L}_{BD} \right\} \tag{2.37}$$

where φ is a scalar field and ω is an arbitrary parameter. $\mathcal{L}_{BD}$ is stipulated to be independent of the scalar field φ, and is related to the matter stress-energy tensor $T^{\mu\nu}$ by

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}} \left(\sqrt{g} \, \mathcal{L}_{BD} \right) , \tag{2.38}$$

just as in general relativity. Since it is only required that the integrand of the action be a scalar, one can in principle have arbitrary functions of φ as coefficients of R and $\varphi_{,\mu} \varphi^{\mu}_{,}$ in (2.37). Thus, Wagoner[23] considered the more general action principle

$$I_W = \int d^4x \{\sqrt{g}\ [h(\varphi) R + \ell(\varphi)\, g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} + 2\lambda(\varphi)] + L_W(\psi^2(\varphi)\, g_{\mu\nu}, \,..)\} \tag{2.39}$$

Here h, ℓ, λ and ψ are arbitrary functions of the scalar field φ. Furthermore, $g_{\mu\nu}$ is not necessarily the metric tensor, but is related to the line element by

$$ds^2 = \psi^2(\varphi)\, g_{\mu\nu}\, dx^\mu\, dx^\nu \tag{2.40}$$

and R is the "scalar curvature" constructed from $g_{\mu\nu}$.

By a representation transformation, Wagoner arrives at the action he prefers to work with:

$$I_W = \int d^4x \left\{\sqrt{\tilde{g}}\ \left[\tilde{R} + n\, \tilde{g}^{\mu\nu}\, \tilde{\varphi}_{,\mu}\, \tilde{\varphi}_{,\nu} + 2\tilde{\lambda}(\tilde{\varphi})\right] + L_W(\tilde{\psi}^2\, \tilde{g}_{\mu\nu}, \ldots)\right\} \tag{2.39a}$$

where

$$\tilde{g}_{\mu\nu} = h\, g_{\mu\nu} \tag{2.41a}$$

$$\left(\frac{d\varphi}{d\tilde{\varphi}}\right)^2 \left[\frac{3}{2} \left(\frac{dh}{d\varphi}\right)^2 + \ell h \right] = nh^2 \tag{2.41b}$$

$$\tilde{\psi}^2(\tilde{\varphi}) = \psi^2 [\varphi(\tilde{\varphi})] / h [\varphi(\tilde{\varphi})] \tag{2.41c}$$

$$\tilde{\lambda}(\tilde{\varphi}) = \lambda [\varphi(\tilde{\varphi})] / h^2 [\varphi(\tilde{\varphi})] \tag{2.41d}$$

$\tilde{\varphi}$ is normalized so that n can be set equal to ± 1. The possibility of $n = 0$ was not considered by Wagoner. However, it will be evident in the following that this possibility must be included for our comparison of the theories. In the representation (2.39a), Wagoner defines the matter stress-energy tensor as

$$G^* T^{\mu\nu} \equiv \frac{2}{\sqrt{\tilde{g}}} \frac{\delta}{\delta \tilde{g}} L_W(\tilde{\psi}^2 \tilde{g}_{\mu\nu}, \cdot\cdot) \tag{2.42}$$

where G^* is a constant. The field equations for both BD and Wagoner theories can be obtained by considering independent variations of the scalar and tensor fields in (2.37) and (2.39a). It can be easily seen, as Wagoner[23] has shown, that the BD theory is a member of the class of theories considered by him.

Disregarding for the moment the content of the matter Lagrangian, we can formally compare the various actions. It is easily seen that (2.30) is formally identical to (2.37) only if the parameter ω takes the numerical value -3/2. Indeed, it was already pointed out by Deser[25] and Anderson[26] that the matter free BD theory is scale invariant only if $\omega = -3/2$. Since (2.30) is by construction scale invariant, the conclusion we have reached regarding the BD parameter ω is not surprising. For a comparison with the Wagoner action it is convenient to make a representation transformation on (2.39) with

$$\tilde{g}_{\mu\nu} = \psi^2 g_{\mu\nu} . \tag{2.43}$$

Thus,

$$I_W = \int d^4x \left\{ \sqrt{\tilde{g}} \left[\frac{h}{\psi^2} \tilde{R} - 6 \tilde{g}^{\mu\nu} \left(\frac{h^{1/2}}{\psi} \right)_{,\mu} \left(\frac{h^{1/2}}{\psi} \right)_{,\nu} \right. \right.$$

$$\left. + h^{-1} \psi^{-2} \left(\frac{3}{2} \left(\frac{dh}{d\varphi} \right)^2 + \ell h \right) \tilde{g}^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} + 2\lambda \psi^{-4} \right] \tag{2.39b}$$

$$\left. + L_W(\tilde{\psi}^2 \tilde{g}_{\mu\nu}, \ldots) \right\}$$

Formal comparison of (2.39b) with (2.30) shows that the Wagoner action can be reduced to our gauge invariant action only if

$$\frac{3}{2} \left(\frac{dh}{d\varphi} \right)^2 + \ell h = 0 \tag{2.44}$$

This in turn demands that $n = 0$ in the Wagoner representation (2.39a). As was pointed out earlier, this case was excluded from Wagoner's[23] considerations.

We remark that if we put $h = -\varphi$, $\ell = -\frac{\omega}{\varphi}$ as required in order to make (2.39) equivalent to (2.37), (2.44) again yields

$$\omega + \frac{3}{2} = 0 , \tag{2.44a}$$

which is the condition for the BD action to formally agree with the gauge invariant action (2.30).

The formal agreement discussed above by no means assures the equivalence of the various theories. When the matter Lagrangian from which the matter stress-energy is defined is taken into account, the non-equivalence of the theories becomes clear. First, by virtue of (2.38) and the fact that $\mathcal{L}_{BD}$ is independent of φ, the BD field equations are consistant with

$$T^{\mu\nu}_{\ ;\nu} = 0 \tag{2.45}$$

the conservation law of general relativity. We have already seen that in the gauge covariant theory (2.45) is no longer valid. Instead, we have the modified conservation law (2.27). Furthermore, Deser[25] has pointed out that if (2.44a) holds the BD field equations would be inconsistant unless the matter stress-energy tensor vanishes. Hence, by construction Brans and Dicke have excluded the possibility of gauge invariance in their theory. Similar remarks can be made about the Wagoner's generalization of the BD theory, even though he does not insist on the validity of (2.45), for it can be easily checked that if $n = 0$, (2.39a) implies $T^{\mu}_{\mu} = 0$ unless ψ is a constant.

It is important to note that in scalar-tensor theories, an independent field equation can be obtained for the scalar field. While (2.30) contains the scalar potential β, no equation is obtained for it. This lack of an independent field equation, which is necessarily the consequence of the imposition of gauge invariance, underlines the difference between Dirac's theory and scalar-tensor theories. In this connection, we remark that Pietenpol et. al.[27] has shown that Dirac's theory can be recast in such a way that the scalar field β is completely decoupled from the other dynamical variables. It is easy to see that the transformation needed to accomplish this is precisely the one which transforms general units into Einstein units, and hence we are back in the domain of general relativity. The possibility of such a transformation is self evident since it has been postulated to begin with. If Dirac's theory is indeed a scalar-tensor theory, such decoupling of the scalar potential from the rest of the dynamics would be a serious difficulty. But in a gauge theory the scalar potential has no dynamical content and is not determined from the dynamics of the theory. Instead, its determination is left as a gauge condition. We feel that an essential point should be emphasized, namely that while in a scale invariant theory gauge transformations give different but equivalent representations of the dynamics, observations would always single out a set of units, and hence a corresponding gauge, so that Einstein units are inadequate if the observing instrument are atomic.

2.5 Conservation Laws

In any action principle, corresponding to coordinate transformation (CT) and gauge transformation (G T) invariance, there are associated conservation laws. In the case of the vacuum, Dirac[7] has already given the details of the derivation of these laws. For CT invariance, one gets the generalized Bianchi identities and for GT invariance, one simply gets an expression which is identically zero.

When matter is present, the GT invariant conservation equation was derived in 2.3. Aside from ensuring that the scalar field equation is not independent, it does not seem to have any sensible physical interpretation. For CT invariance, one can proceed formally as indicated by Dirac[7]. After some tedious algebra, one arrives at precisely the conservation equation (2.27). Instead of producing all the details of this derivation, we shall pursue (2.27) further by introducing the energy momentum tensor of a perfect fluid,

$$T^{\mu\nu} = (\rho + p)\, u^{\mu} u^{\nu} - p\, g^{\mu\nu} \tag{2.46}$$

Projection of (2.27) parallel and orthogonal to u^{μ}, yields respectively the energy equation

$$\dot{\rho} + (\rho + p)\, u^{\mu}{}_{;\mu} = -\rho \left(\frac{\dot{G}}{G} + \frac{\dot{\beta}}{\beta} \right) - 3p\, \frac{\dot{\beta}}{\beta} \tag{2.47}$$

and the Euler equation

$$(\rho + p)\, \dot{u}^{\mu} = (g^{\mu\nu} - u^{\mu} u^{\nu}) \left(p_{,\nu} + p\, \frac{G_{,\nu}}{G} + (\rho - p)\, \frac{\beta_{,\nu}}{\beta} \right) \tag{2.48}$$

For a co-moving volume $V = R^3$, an alternative form of (2.47) is

$$\frac{(\rho V G \beta)^{\cdot}}{(\rho V G \beta)} = \frac{p}{\rho} \frac{(R \beta)^{\cdot}}{(R \beta)} \tag{2.47a}$$

We recall that the gravitational "constant" is now a function of space-time and its derivatives do not vanish in general. Equation (2.47) and (2.48) show explicitly how the variation of G and β modifies the energy and momentum conservation laws of general relativity when written in general units.

Next we consider yet another conservation equation whose physical content is not contained in the action principle. In hydrodynamic problems encountered in general relativity, it is necessary to have an equation for the number density of particles in order for the system of hydrodynamic equations to be closed. The baryon number density is also conserved. The differential form of this conservation law is written as

$$(\bar{n}\ \bar{u}^{\mu})_{;\mu} = 0 \tag{2.49}$$

If one further assumes that the particle rest mass $\bar{m}$ is constant, the above equation can be expressed as

$$(\bar{\mathfrak{M}}\ \bar{u}^{\mu})_{;\mu} = 0 \tag{2.50}$$

where

$$\overline{\mathfrak{M}} = \overline{m}\ \overline{n}$$

is the rest mass density. When the internal energy of the system is sufficiently low so that no pair creation or annihilation occurs, such as in classical fluid dynamics, (2.50) is known as the mass conservation law.

Since Dirac's LNH raises the possibility of non-conservation of baryonic number, we shall seek a generalization of (2.49) or (2.50) so that baryon number is not necessarily a conserved quantity. We assume that (2.50) is valid in Einstein units. Generalization of (2.50) according to our co-covariant considerations can be simply written as (see (A1.16))

$$\begin{aligned}(\mathfrak{M}\, u^{\mu})_{*\mu} &= (\mathfrak{M}\, u^{\mu})_{;\mu} - (\Pi + 4)\, \mathfrak{M}\, u^{\mu}\, k_{\mu} \\ &= 0\end{aligned} \tag{2.51}$$

The co-tensor power Π of $\mathfrak{M}\ u^{\mu}$ can be deduced as follows. First, it is clear from its definition that u^{μ} has power -1. $\mathfrak{M}$ is the classical limit of ρ, the energy density and hence has the same power as ρ, which we denote by $\Pi(\rho)$. Furthermore, we denote the co-scalar power of G by $\Pi(G)$. From the knowledge that $*\mathfrak{J}_{\mu\nu}$ is an in-tensor, we can write (see 2.24)

$$\Pi(G) + \Pi(\rho) = -2 \tag{2.52}$$

so that

$$\Pi \equiv \Pi(\mathfrak{M} u^{\mu}) = \Pi(\mathfrak{M}) + \Pi(u^{\mu})$$

$$= \Pi(\rho) - 1$$

$$= - \Pi(G) - 3$$

Consequently, equation (2.51) becomes

$$(\mathfrak{M} u^{\mu})_{;\mu} - [\Pi(G) - 1]\, \mathfrak{M} \frac{\dot{\beta}}{\beta} = 0 \tag{2.51a}$$

where equation (2.25) has been used in the above reduction. In atomic units, particle rest mass is constant and we obtain an equation for the particle number density

$$(n u^{\mu})_{;\mu} - [\Pi(G) - 1] n \frac{\dot{\beta}}{\beta} = 0 \tag{2.53}$$

$\Pi(G)$ cannot be specified independently of the gauge condition. Examples of its determination will be given in the next section. Finally, we note that the assumption of validity of (2.50) in Einstein units and its consequences in (2.51a) and (2.53) is consistant with our previous treatment of the gauge covariant field equations. It is easy to see that in the classical limit, when $p = 0$, (2.47) is equivalent to (2.51a). But we emphasize that (2.51a) is an independent equation since it is assumed to be valid even when matter pressure is non-vanishing.

2.6 LNH as a Gauge Condition

Since we do not yet know the functional form of G or β, we have only the formal structure of a theory. To be able to solve dynamical problems, we must specify β which corresponds to choosing a gauge. We shall now give an example of how the LNH can be used to specify β in cosmology.

G is a co-scalar, and we assume it has power $\Pi(G)$. In Einstein units, it has a constant value $\overline{G}$. β has been shown to be a co-scalar of power -1. In Einstein units it is a constant which we can set equal to unity. Thus, generally, we can write

$$G = \overline{G}\ \beta^{-\Pi(G)} \sim \frac{1}{t} \tag{2.54}$$

where the second relation results from a consequence of the LNH, namely that the gravitational constant in atomic units is inversely proportional to the cosmological time. We next consider (2.51a) in a cosmological context. Equation (2.51a) implies for a co-moving volume $\mathcal{V}$

$$\frac{1}{\mathcal{V}}\ (\mathfrak{M}\mathcal{V})^{\cdot} = [\Pi(G) - 1]\ \mathfrak{M}\ \frac{\dot{\beta}}{\beta}$$

or

$$(\mathfrak{M}\mathcal{V}\ G\beta)^{\cdot} = 0$$

a result to be expected as a particular case of (2.47a) since when $p \to 0$, $\rho \to \rho_0 = \mathfrak{M}$ (rest mass density). We therefore have

$$\mathfrak{M}\mathcal{V} \sim G^{-1}\ \beta^{-1} \sim \beta^{(\Pi(G) - 1)} \sim t^2 \tag{2.55}$$

where the second relation states that the mass in a co-moving element increases like the square of cosmological time, which is another consequence of LNH. Combining (2.54) and (2.55), we find

$$\beta \sim \frac{1}{t} \tag{2.56}$$

and

$$\Pi(G) = -1 \tag{2.57}$$

$$G = \overline{G}\,\beta \tag{2.58}$$

On the other hand, if we do not assume spontaneous mass creation and require

$$\mathfrak{M}\,V \sim \beta^{\Pi(G) - 1} \sim t^{0} \tag{2.55a}$$

we obtain instead

$$\beta \sim t \tag{2.56a}$$

$$\Pi(G) = 1 \tag{2.57a}$$

$$G = \overline{G}\,\beta^{-1} \tag{2.58a}$$

Thus we see that the LNH can be considered the observational input which determines the atomic gauge relative to the Einstein gauge, and therefore the function $\beta(t)$. With this known functional form of β and hence known variation of G, the field equations derived earlier are complete and solutions can then be obtained to yield various cosmological models just as in general relativity.

It should be remarked that we have not used LNH in the general form (A2.5). Rather, we are considering (A2.4a) and (A2.4b) as separate hypotheses which can be adopted in conjunction or separately. Our purpose is to use relations of the type (A2.4) to determine the gauge condition. Whether (A2.5) is consistant with the gauge covariant theory can then be subjected to tests using the dynamical equations. It is interesting to note that if both (A2.4a) and (A2.4b) are used in the determination of β, (A2.6) follows as will be shown in the following.

III. Cosmology

Having determined the atomic gauge and hence a specific functional form for β, the dynamical equations can be applied for cosmological considerations. We shall assume, as in the standard cosmological model, spatial homogeneity and isotropy. The line element can be written as

$$ds^2 = dt^2 - R^2(t)\ \gamma_{ij}\ dx^i\ dx^j \tag{3.1}$$

where γ_{ij} is the spatial part of this metric and is not a function of t. With appropriate coordinates, (3.1) can be written in Robertson-Walker form

$$ds^2 = dt^2 - R^2(t) \left[\frac{dr^2}{1-kr^2} + r^2\, d\theta^2 + r^2 \sin\theta\, d\varphi^2 \right] \tag{3.1a}$$

where k is a parameter which can be normalized to ± 1 or 0. We keep in mind that ds is in atomic units. When the occasion arises, quantities in Einstein units will be indicated by a bar over the symbol as has been done in §II.

From (3.1), the field equations (2.34) become

$$\left(\frac{\dot{R}}{R} + \frac{\dot{\beta}}{\beta} \right)^2 + \frac{k}{R^2} = \frac{8\pi}{3} G\rho + \frac{1}{3}\Lambda \tag{3.2a}$$

$$\frac{\ddot{R}}{R} + \frac{\ddot{\beta}}{\beta} + \frac{\dot{\beta}}{\beta}\frac{\dot{R}}{R} - \frac{\dot{\beta}^2}{\beta^2} = -\frac{4\pi G}{3}(3p + \rho) + \frac{1}{3}\Lambda \tag{3.2b}$$

As in ordinary cosmology, the dynamic equation (3.2a) must be supplemented by the "energy conservation" equation (2.47)

$$\dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + p) = - \rho \frac{(G\beta)^{\bullet}}{G\beta} - 3p \frac{\dot{\beta}}{\beta}$$

which, for any equation of state of the form

$$p = c_s^2 \rho$$

can be integrated to give

$$G(\beta) \rho(\beta) R^{3(1 + c_s^2)} \sim \frac{1}{\beta^{1 + 3 c_s^2}} \tag{3.3}$$

Specifically [2.57, 2.57a]

$$\rho R^{3(1 + c_s^2)} \sim \begin{cases} \beta^{-2 - 3 c_s^2} & \text{A) matter creation} \\ \beta^{-3 c_s^2} & \text{B) no matter creation} \end{cases} \tag{3.4}$$

For dust ($c_s^2 = o$) and radiation ($c_s^2 = 1/3$) the previous equations specialize to

$$\text{A)}\quad \rho_m(t) \sim \frac{1}{\beta^2(t)}\, R^{-3}(t) \qquad ; \qquad \text{B)}\quad \rho_m(t) \sim R^{-3}(t) \tag{3.5a}$$

$$\text{A)}\quad \rho_\gamma(t) \sim \frac{1}{\beta^3(t)}\, R^{-4}(t) \qquad ; \qquad \text{B)}\quad \rho_\gamma(t) \sim \frac{1}{\beta(t)}\, R^{-4}(t) \tag{3.5b}$$

Equation (3.5) can also be obtained directly by gauge transforming the corresponding results in Einstein units, where as we know

$$\begin{aligned} \bar{\rho}_m(t) &\sim \left(\frac{\Re(t_o)}{\Re(t)}\right)^3 \\ \bar{\rho}_\gamma(t) &\sim \left(\frac{\Re(t_o)}{\Re(t)}\right)^4 \end{aligned} \tag{3.6}$$

For either case (dust or radiation) the relations between atomic and Einstein density can be written as

$$\rho(t) = \bar{\rho}(\tau)\ \beta^{-\Pi(\rho)} \tag{3.7}$$

i.e. the atomic density $\rho(t)$ has an a priori unknown dependence on β that we parameterize as $\beta^{-\Pi(\rho)}$ i.e. $\rho(t)$ is a co-scalar of power $\Pi(\rho)$. However because of (2.52), we can rewrite (3.7) as

$$\rho(t) = \bar{\rho}(t)\ \beta^{\Pi(G)+2} \tag{3.8}$$

with $\Pi(G) = \pm 1$ as from (2.57)-(2.58). Remembering also that by definition

$$\Re(t) = R(t)\ \beta(t) \tag{3.9}$$

we finally obtain

$$\begin{aligned} \rho_m(t) &= \beta_o^3\ \bar{\rho}_{mo}\ \frac{1}{\beta^{1-\Pi(G)}}\left(\frac{R(t_o)}{R(t)}\right)^3 \\ \rho_\gamma(t) &= \beta_o^4\ \bar{\rho}_{\gamma o}\ \frac{1}{\beta^{2-\Pi(G)}}\left(\frac{R(t_o)}{R(t)}\right)^4 \end{aligned} \tag{3.3}$$

which coincide with (3.3). In the same spirit, we shall now present an alternative derivation of the two Einstein equations (3.2). In Einstein's units we have

$$\begin{aligned} \dot{\Re}^2(\tau) + k &= \frac{8\pi\bar{G}}{3}\ \bar{\rho}\ \Re^2(\tau) + \tfrac{1}{3}\bar{\Lambda}\ \Re^2 \\ \ddot{\Re}(\tau) &= -\frac{4\pi\bar{G}}{3}\ (3\bar{p} + \bar{\rho})\ \Re + \tfrac{1}{3}\bar{\Lambda}\ \Re \end{aligned} \tag{3.10}$$

Using (3.9) and the fact that

$$\beta(\tau)\, dt = d\tau \tag{3.11}$$

we have

$$\dot{\mathfrak{R}}(\tau) = \frac{d\,\mathfrak{R}(\tau)}{d\tau} = \frac{d\,[R(t)\,\beta(t)]}{\beta(t)\,dt} = R(t)\left[\frac{\dot{\beta}(t)}{\beta(t)} + \frac{\dot{R}(t)}{R(t)}\right]$$

$$\ddot{\mathfrak{R}}(\tau) = \frac{d^2\,\mathfrak{R}(\tau)}{d\tau^2} = \frac{R(t)}{\beta(t)}\left[\frac{\ddot{R}(t)}{R(t)} + \frac{\ddot{\beta}}{\beta} + \frac{\dot{\beta}}{\beta}\,\frac{\dot{R}}{R} - \frac{\dot{\beta}^2}{\beta^2}\right] \tag{3.12}$$

which upon insertion in (3.10) yield (3.2) if we put

$$p\,G = \beta^{-2}\,\bar{G}\,\bar{p}\;, \qquad \Lambda = \bar{\Lambda}\,\beta^2$$

$$\rho\,G = \beta^{-2}\,\bar{G}\,\bar{\rho} \tag{3.13}$$

Using (3.3) to eliminate $G(\beta)\ \rho(\beta)$, we can now solve Eq. (3.2), which we shall write as

$$\frac{dF}{\beta\, dt} = \frac{dF}{d\tau} = \left(\frac{a}{F} + \frac{1}{3}\,\bar{\Lambda}\,F^2 - k\right)^{1/2} \tag{3.14}$$

where

$$F = R(t)\,\beta(t)\;, \qquad a \equiv \frac{8\pi\,\bar{G}}{3}\,\rho_{mo}\,R_o^3$$

Upon integrating, we find

$$\tau = \int^{F/F_o} \frac{dF}{\sqrt{\frac{a}{F} - k + \frac{1}{3}\bar{\Lambda} F^2}} \tag{3.15}$$

a well-known equation in ordinary cosmology. For $k = \pm 1,\ 0$, the solutions are respectively $(\Lambda = 0)$

$$\begin{aligned}
&F(\tau) \sim 1 - \cos\theta\ , \quad \tau \sim \theta - \cos\theta && k = +1 \\
&F(\tau) \sim \tau^{2/3} && k = 0 \\
&F(\tau) \sim \cosh\psi - 1\ , \quad \tau \sim \sinh\psi - \psi && k = -1
\end{aligned} \tag{3.16}$$

For large τ, we obtain

$$F(\tau) \sim \tau^{2/3}\ , \quad F(\tau) \sim \tau^{2/3}\ , \quad F(\tau) \sim \tau$$

Translating back into atomic units, the $R(t)$ vs t functions now read

A) $\beta = t_o/t$,

$$\begin{aligned}
&R(t) \sim t\,(\ell n\ t)^{2/3} && k = +1 \\
&R(t) \sim t\,(\ell n\ t)^{2/3} && k = 0 \qquad (q_o \ll 1) \\
&R(t) \sim t\ \ell n\ t && k = -1
\end{aligned} \tag{3.17}$$

B) $\beta = t/t_o$;

$$R(t) \sim t^{1/3} \qquad k = +1 , \qquad q_o = 2$$

$$R(t) \sim t^{1/3} \qquad k = 0 , \qquad q_o = 2 \tag{3.18}$$

$$R(t) \sim t \qquad k = -1 , \qquad q_o = 0$$

We must stress that the k = 0 case corresponds to an exact solution. For the case with $\Lambda \neq 0$, we can only give an asymptotic solution. It is clear from (3.14) that at large enough times the right hand side of (3.14) goes like $(\bar{\Lambda}/3)^{1/2}$ F an so the solution is

$$F(\tau) \sim \exp \sqrt{\frac{\bar{\Lambda}}{3}} \; \tau$$

or

$$R(t) \sim t \exp\left(\sqrt{\frac{\bar{\Lambda}\, t_o^2}{3}} \; \ell n \; t \right) \sim t^{1+a} , \quad (a << 1) \tag{3.19}$$

Equations (3.17), (3.18), (3.19) represent the main result of the gauge-covariant theory as applied to cosmology.

The final step necessary to make the presentation complete concerns the derivation of the relation between k, ρ_o and H_o. Introducing the notation

$$q_o = -\left(\frac{R\ddot{R}}{\dot{R}^2}\right)_o \quad , \quad H_o = \left(\frac{\dot{R}}{R}\right)_o$$

$$Q_o = -\left(\frac{\beta\ddot{\beta}}{\dot{\beta}^2}\right)_o \quad , \quad h_o = \left(\frac{\dot{\beta}}{\beta}\right)_o \tag{3.19}$$

we can easily derive from (3.2) the following relations

$$\frac{\rho}{\rho_c} = 2q_o + \frac{2\Lambda}{3H_o^2} + 2(1+Q_o)\frac{h_o^2}{H_o^2} - 2\left(\frac{h_o}{H_o}\right)$$

$$\frac{k}{R_o^2} = (2q_o - 1)H_o^2 + \Lambda + (1+2Q_o)h_o^2 - 4h_o H_o \tag{3.20}$$

or

$$\frac{k}{R_o^2} = \left\{\frac{\rho}{\rho_c} - \left(1 + \frac{h_o}{H_o}\right)^2\right\}\left(\frac{H_o}{c}\right)^2 + \frac{\Lambda}{3}$$

a generalization of the well-known relations in Einstein units

$$\frac{\bar{\rho}}{\bar{\rho}_c} = 2\bar{q}_o + \frac{2\bar{\Lambda}}{3\bar{H}_o^2} \quad , \quad \frac{k}{\bar{R}_o^2} = (2\bar{q}_o - 1)\bar{H}_o^2 + \bar{\Lambda}$$

$$\frac{k}{R_o^2} = \left(\frac{\bar{\rho}}{\bar{\rho}_c} - 1\right)\left(\frac{\bar{H}_o}{c}\right)^2 + \frac{\Lambda}{3} \tag{3.21}$$

The relation between $\bar{q}_c$ and q_o is derived to be

$$\bar{q}_o = q_o \left(\frac{H_o}{h_o + H_o} \right)^2 + (1 + Q_o) \left(\frac{h_o}{h_o + H_o} \right)^2 - \frac{h_o H_o}{(h_o + H_o)^2}$$

$$\bar{H}_o = H_o + h_o \, , \quad \bar{\rho}_c = \rho_c \, \beta_o \left(\frac{H_o + h_o}{H_o} \right)^2 \tag{3.22}$$

At this point we must discuss a very important point concerning q_o and $\bar{q}_o$. In traditional cosmology the search for the value of the curvature has been pursued in the last sixteen years by Sandage and his collaborators. Recently however the abundance of deuterium, an element very difficult to form but in the early universe, has proved to be a more sensitive test than any of the ones used by Sandage so far. The conclusion based on the abundance of deuterium is that the universe is open and the value of the deceleration parameter is much less than unity. This value is often mistakenly identified with q_o, thus ruling out the $k = +1, 0$ cases for $\beta = t/t_o$. However this is not right. The experimental value should be identified with $\bar{q}_o$, because the cosmological models employed in the nucleosynthesis computations done so far correspond to Einstein units with $\Lambda = 0$. Knowing the experimental value of $\bar{q}_o$ we can insert it in (3.22) and upon using (3.17) and (3.18) evaluate the right hand side and check for consistency.

The first case to be considered will be the one corresponding to no-matter creation (3.18). Since in the three cases the R(t) function can be written as t^a ($a = 1/3$ or 1), it is easy to check that (3.22) becomes

$$\bar{q}_o = \frac{1-a}{1+a} \quad , \quad q_o = \frac{1-a}{a} > 0$$

$$\bar{q}_o = q_o \frac{a}{1+a} \tag{3.23}$$

The value of $\bar{q}_o$ is clearly less than one for any of the three curvatures $k = \pm 1, 0$; in particular $\bar{q}_o = .5$ for $k = +1, 0$ and $\bar{q}_o = 0$ for $k = -1$, whereas q_o is zero or two. This clearly indicates how incorrect it is to compare q_o instead of $\bar{q}_o$ with observations.

The case A) can also be treated. By writing $R = t(\ell n\, t)^b$, ($b = 2/3$ or 1), thus encompassing the three cases, it is easy to derive that

$$\bar{q}_o = \frac{1-b}{b} \; ; \quad - q_o = \frac{b\, \ell n\, t + b(b-1)}{(b + \ell n\, t)^2} \tag{3.24}$$

Here again, as before, $\bar{q}_o < 1$ and in particular $\bar{q}_o = .5$ for $k = +1$ and 0 and $\bar{q}_o = 0$ for $k = -1$. This completes our exposition of the cosmological consequences of the gauge - covariant theory of gravitation. As will be shown later, case A), corresponding to matter creation, seems at present favorable over B). In this case we would suggest that the $k = -1$ curvature case, with $R \sim t\, \ell n\, t$, is more likely to be the model that best fits the cosmological data in that it yields the smallest value of $\bar{q}_o$, namely zero, as the growing evidence from the abundance of deuterium seems to indicate. At the level of numerical coincidences and Mach principle

$$\frac{MG}{Rc^2} \sim \text{const.} \quad , \tag{3.25}$$

Dirac has suggested that $R \sim t$, for the matter creation case. In fact if $M \sim t^2$, $G \sim t^{-1}$, R must go like t. Clearly such a behavior is only reproduced by the $k = -1$ curvature case, without matter creation however. Within the matter creation case it is clear that a pure $R \sim t$ is not an admissible solution for the $\Lambda = 0$ case. However it ought to be remembered that the Mach principle (3.25) is actually not incorporated into the set of Einstein equations (3.2) and so its use corresponds to an extra boundary conditions. This is most clearly seen if we write (3.25) as

$$\frac{\rho G}{R^2} \sim \text{const.} \tag{3.26}$$

The product ρG is a co-scalar of power -2 [(2.52) and (3.13)] and so

$$\frac{\bar{\rho}\,\bar{G}}{(\beta R)^2} = \text{const.} \tag{3.27}$$

This implies $\beta R = \Re = \text{const.}$, i.e. we must have a static Einstein universe in Einstein units. Case B), with $\beta \sim t$, is evidently excluded since it would imply $R \sim 1/t$ i.e. a contracting universe, a fact against all existing evidence. For Case A) with matter creation, $\beta \sim \frac{1}{t}$, we have $R \sim t$, i.e. the universe expands, an admissible solution. In this case the cosmological constant Λ must be different from zero as is clear from (3.10)

$$\bar{\Lambda} = 4\pi \bar{G} \bar{\rho} = \frac{k}{\bar{\Re}^2} \tag{3.28}$$

which in turn implies $k = +1$. Clearly such a model is not based on a very credible basis. Mach principle as expressed by (3.25)-(3.27) is imposed upon the equations, it does not come out of them in a natural way and for all we know it could even have, should we ever be able to derive it from first principles, a function $\beta(t)$ attached to it that would alter (3.27). We therefore prefer to stick to the exact solutions represented by (3.17), (3.18) and (3.19) without postulating any additional external boundary condition. Finally we would like to comment on the existence of the large number N_3 [see Appendix A2.3]

$$N_3 = \frac{4\pi}{3} \frac{\rho}{m_p} \left(\frac{c}{H_o}\right)^3 \approx 10^{78} \sim t^2 \tag{3.29}$$

By asserting that (3.29) should hold for all cosmological times, Dirac concluded that one must require matter creation. But in the construction of the large number N_3, the present expansion parameter H_o was used to define the visible universe, whose coordinate boundaries may change with time. Hence the variation of N_3 with time need not imply matter creation. In fact by using $\rho_m(t)$ from case B), (3.5a), corresponding to non-matter creation, $\rho_m \sim 1/R^3(t)$ and (3.18) for either $k = +1$ or zero, the quantity

$$N_3 \sim \frac{\rho(t)}{H^3(t)} \simeq \frac{1}{R^3(t)} \left(\frac{R(t)}{\dot{R}(t)}\right)^3 \sim \frac{1}{\dot{R}^3(t)} \sim t^2$$

goes exactly like t^2 and no matter creation is needed. Granting that the LNH can be meaningfully used to fix the gauge function $\beta(t)$, we must emphasize that the cosmological solutions presented here are valid only for large cosmological times and cannot be extrapolated to early times. If one does so,[14] one finds that the mean free time for nuclear interactions as well as the mean free time for photon compton scattering are greater than the expansion time of the universe itself and therefore no nucleosynthesis could have taken place.

As repeatedly stressed by Dirac, the LNH is an asymptotic condition and it cannot be used to fix the value of $\beta(t)$ at times when nucleosynthesis occurred. A new condition must be found. For exactly the same reason we cannot at this moment make any sensible comment on the existence of an horizon, since that again implies the knowledge of the function $R(t)$ and therefore $\beta(t)$ for any t.

IV. Application to Local Gravitational Phenomena

4.1 Equations of Motion

It is well-known that the general theory of relativity predicts an advance in perihelion for bound orbits. In the gauge-covariant theory, the epochal variations of the gauge field β and the combination GM, which were found to obey the law

$$GM\beta = \text{constant} \tag{2.55}$$

where

$$M = \mathfrak{M}V$$

will cause further secular changes in the orbital elements. While these changes are of interest per se, most important in this section, however, will be the application of these results to the three classic tests: the advance of perihelion, the deflection of light and the radar echo delay.

We begin by writing the generalized geodesic equations (2.28), namely

$$u^{\mu}{}_{;\nu}\, u^{\nu} = (\epsilon\, g^{\mu\nu} - u^{\mu} u^{\nu}) \frac{\beta_{,\nu}}{\beta} \tag{4.1}$$

where

$$\begin{aligned} \epsilon &= 0 \quad \text{for photons} \\ \epsilon &= 1 \quad \text{for particles} \end{aligned} \tag{4.2}$$

and $u^{\mu} = \frac{dx^{\mu}}{dp}$, p a parameter along the geodesic. Since β can vary with time either as t^{-1} for the matter creation case or as t, for the case without matter creation, we shall write in general

$$\beta(t) = \left(\frac{t}{t_o}\right)^{\eta} , \quad \eta = \pm 1 \tag{4.3}$$

The absence of disturbances perpendicular to the ecliptic allows us to characterize this plane by $\theta = \frac{\pi}{2}$. In this case the Schwarzschild metric is

$$ds^2 = B(r,t)dt^2 - A(r,t)dr^2 - r^2 \, d\varphi^2 \tag{4.4}$$

where

$$B = A^{-1} = 1 - 2\frac{GM}{r} \tag{4.5}$$

with

$$\dot{A} = -\frac{1}{B^2} \dot{B} , \quad \dot{B} = \frac{2\,GM}{r} \frac{\dot{\beta}}{\beta} \tag{4.6}$$

In cylindrical co-ordinates, the geodesic equations (4.1) are

$$\frac{d^2 r}{dp^2} - \frac{r}{A}\left(\frac{d\varphi}{dp}\right)^2 + \frac{1}{2A}\frac{\partial B}{\partial r}\left(\frac{dt}{dp}\right)^2 + \frac{1}{2}\frac{1}{A}\frac{\partial A}{\partial r}\left(\frac{dr}{dp}\right)^2 + \frac{dr}{dp}\frac{dt}{dp}\left(\frac{\dot{\beta}}{\beta} + \dot{A}\right) = 0 \tag{4.7}$$

$$\frac{d^2\varphi}{dp^2} + \frac{2}{r}\frac{d\varphi}{dp}\frac{dr}{dp} + \frac{dt}{dp}\frac{d\varphi}{dp}\frac{\dot{\beta}}{\beta} = 0 \tag{4.8}$$

$$\frac{d^2 t}{dp^2} + \frac{1}{B}\frac{\partial B}{\partial r}\frac{dr}{dp}\frac{dt}{dp} + \frac{1}{2B}\dot{A}\left(\frac{dr}{dp}\right)^2 + \frac{1}{2}\dot{B}\left(\frac{dt}{dp}\right)^2 = \left[\frac{\epsilon}{B} - \left(\frac{dt}{dp}\right)^2\right]\frac{\dot{\beta}}{\beta} \tag{4.9}$$

Rather than equation (4.7), it is more convenient to use the relation

$$B\left(\frac{dt}{dp}\right)^2 - A\left(\frac{dr}{dp}\right)^2 - r^2\left(\frac{d\varphi}{dp}\right)^2 = \epsilon \tag{4.10}$$

which paraphrases

$$g_{\mu\nu}\, u^{\mu}\, u^{\nu} = \epsilon \tag{4.11}$$

which in turn follows directly from equation (4.1). By defining new variables E and J as

$$E = B\,\frac{dt}{dp} \tag{4.12}$$

$$J = r^2 \frac{d\varphi}{dp} \tag{4.13}$$

equations (4.8) and (4.9) may be written in the equivalent forms

$$\frac{d}{dp} J\beta = 0 \tag{4.14}$$

and

$$\frac{E}{B} \frac{d}{dp} \log E = - \frac{1}{2B} \dot{A} \left(\frac{dr}{dp}\right)^2 + \frac{1}{2} \left(\frac{E}{B}\right)^2 \frac{\overset{\circ}{B}}{B} + \left[\frac{\varepsilon}{B} - \left(\frac{E}{B}\right)^2\right] \frac{\dot{\beta}}{\beta} \tag{4.15}$$

In the standard theory, where $\beta = 1$ and A, B are independent of t, E and J are the conserved energy and angular momentum respectively. The gauge-covariant theory induces secular changes in E and J. Eliminating the variable p, we arrive at the equations of motion parameterized by the time t,

$$\frac{E^2}{B} - A \frac{E^2}{B^2} \left(\frac{dr}{dt}\right)^2 - \frac{J^2}{r^2} = \epsilon \tag{4.16}$$

$$J\beta = r^2 \frac{d\varphi}{dt} \frac{E}{B} \beta = J_o = \text{constant} \tag{4.17}$$

and

$$\frac{\dot{E}}{E} = - \frac{\dot{A}}{2B} \left(\frac{dr}{dt}\right)^2 + \frac{1}{2} \frac{\dot{B}}{B} + \left(\frac{\varepsilon B}{E^2} - 1\right) \frac{\overset{\circ}{\beta}}{\beta} \tag{4.18}$$

The equation for the trajectory is obtained by eliminating the variable t between (4.16) and (4.17),

$$\frac{1}{r^4}\left(\frac{dr}{d\varphi}\right)^2 = \left(\frac{E}{J}\right)^2 - B\left(\frac{\varepsilon}{J^2} + \frac{1}{r^2}\right) \tag{4.19}$$

The equations of motion derived so far apply equally well to particles and photons, depending on the choice of ϵ.

4.2 Perihelion Advance [29]

Since E varies slowly with time, we may replace the instantaneous value in (4.19) by its average over one revolution, $\bar{E}$. The variable J, defined by (4.17) is also a slowly varying function. Thus

$$\frac{1}{r^4}\left(\frac{dr}{d\varphi}\right)^2 = \left(\frac{\bar{E}}{J}\right)^2 - B\left(\frac{1}{J^2} + \frac{1}{r^2}\right) \tag{4.20}$$

Since equation (4.20) is formally identical to the standard relativistic orbit equation, we can immediately write down the result for the perihelion advance per revolution for a quasi-elliptic orbit, viz.

$$\Delta\varphi = \frac{6\pi MG}{L} \tag{4.21}$$

where L is the semi-latus rectum, related to the ap- and peri-centers r_+ and r_- of the ellipse by the formula

$$\frac{1}{L} = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right) \tag{4.22}$$

If by a and e we denote the semi-major axis and eccentricity, we have

$$\begin{aligned} L &= a(1-e^2) \\ r_\pm &= a(1 \pm e) \end{aligned} \tag{4.23}$$

and our aim is to connect a and e to $\overline{E}$ and J, whose evolution with time is known.

The peri- and ap-centers are given by setting

$$\frac{dr}{d\varphi} = 0$$

in (4.20), from which it follows that

$$\frac{\overline{E}^2}{B(r_\pm)} - \frac{J^2}{r_\pm^2} = 1 \tag{4.24}$$

Solving for $\overline{E}$ and J, we get

$$J^2 = \frac{B(r_-) - B(r_+)}{\dfrac{B(r_+)}{r_+^2} - \dfrac{B(r_-)}{r_-^2}} \tag{4.25}$$

and

$$\bar{E}^2 = \frac{r_+^2 - r_-^2}{\frac{r_+^2}{B(r_+)} - \frac{r_-^2}{B(r_-)}} \tag{4.26}$$

The definitions (4.23) enable us to write, to first order in GM/a

$$J^2 = GM\,a\,(1-e^2)\left(1 + \frac{GM}{a}\,\frac{3+e^2}{1-e^2}\right) \tag{4.27}$$

and

$$\bar{E}^2 = 1 - \frac{GM}{a} \tag{4.28}$$

From (4.27), we find

$$\left(\frac{J}{GM}\right)^2 = \left(\frac{J_o}{\beta\,GM}\right)^2 = \frac{L}{GM}\left(1 + \frac{3+e^2}{L}\,GM\right) \tag{4.29}$$

To first order in GM/L, we therefore find that

$$\Delta\varphi = \frac{6\pi\,GM}{L} = 6\pi\left(\frac{\beta\,GM}{J_o}\right)^2 \tag{4.30}$$

which is a constant. That is, the perihelion advance per revolution is independent of time. Since GM is proportional to β^{-1}, we conclude that to first order in GM/L, the precession has the same value as in standard theory.

For future reference, we shall now deduce how the individual elements e and a vary with time. Their behavior is governed by equation (4.18). As we are interested only in near-Newtonian Keplerian orbits, we set

$$E = 1 - E' \tag{4.31}$$

and so by equation (4.28)

$$E' = \frac{GM}{2a} \tag{4.32}$$

Equation (4.18) then becomes

$$-\frac{dE'}{dt} = -\frac{B}{2E^2}\dot{A}\left[E^2 - B\left(\frac{J^2}{r^2}+1\right)\right] + \frac{1}{2}\frac{\dot{B}}{B} + \left(\frac{B}{E^2}-1\right)\frac{\dot{\beta}}{\beta} \tag{4.33}$$

Using the expressions for $\dot{A}$ and $\dot{B}$ (4.6), we finally arrive at

$$\frac{dE'}{dt} = \frac{\dot{\beta}}{\beta}\left[\frac{GMJ^2}{r^3} + \frac{GM}{r} + 2E'\right] \tag{4.34}$$

Averaging over one period, we find

$$\frac{d\overline{E'}}{dt} = \frac{\dot{\beta}}{\beta} \frac{GMJ^2}{a^3(1-e^2)^{3/2}} \tag{4.35}$$

where we have used the following averages over a Keplerian orbit

$$\begin{aligned} \overline{r^{-3}} &= \frac{1}{a^3(1-e^2)^{3/2}} \\ \overline{r^{-1}} &= \frac{1}{a} \end{aligned} \tag{4.36}$$

Further manipulation of (4.35) leads to

$$\frac{d\overline{E'}}{dt} = \frac{\eta}{t} \frac{GM}{J} (2E')^{3/2} \tag{4.37}$$

whose quadrature yields

$$\overline{E'} = \frac{\overline{E'_o}}{\left(1 - (2\,\overline{E'_o})^{1/2} \frac{GM}{J} \log \beta\right)^2} \tag{4.38}$$

where $\overline{E}'_o = \left(\frac{GM}{2a}\right)_o$ is the total energy of system today. The expressions for a and e follow immediately from (4.32), (4.29) and (4.38).

$$\frac{a}{a_o} = \beta^{-1} \left[1 - \left(\frac{GM}{a(1-e^2)^{1/2} c^2} \right)_o \log \beta \right]^2 \tag{4.39}$$

$$1 - e^2 = \frac{(1 - e^2)_o}{\left[1 - \left(\frac{GM}{a(1-e^2)^{1/2} c^2} \right)_o \log \beta \right]^2} \tag{4.40}$$

We conclude that the semi-major axis and the eccentricity both vary monotonically with time.

4.3 Radar Echo Delay[29]

For several years Shapiro and his co-workers have been engaged in measuring the delay in the radar signal broadcast in the direction of Mercury and bounced back. The delay is caused by the apparent slowing down of the speed of light near the rim of the sun during grazing incidence. Here we are interested in changes in the result of this experiment when carried out over a long period. The changes come from two different, though not unrelated causes. First, had the planets (Earth and Mercury) remained fixed in their tracks, the gauge-covariant theory predicts a slow variation in successive time delay measurements; second, the fact that the orbital elements themselves change (as has been shown in 4.2) means that the light ray has to span different Earth-Mercury distances as time goes on. It will be shown that the first kind of variation goes like β^{-1}, and the second kind of variation enters only almost as β^{-1}, being modified by a logarithmic term.

Since the photon is massless, the energy and angular momentum per unit mass become infinite. The ratio $b = J/E$ remains finite, however, and is what is known in the standard theory as the impact parameter. The equations of motion are then

$$\frac{A}{B^2}\left(\frac{dr}{dt}\right)^2 + \frac{b^2}{r^2} - \frac{1}{B} = 0 \tag{4.41}$$

$$\frac{1}{b}\ \frac{db}{dt} = -\ \frac{GM}{Br\,t}\ \eta \tag{4.42}$$

To evaluate the approximate changes in b, we need to integrate (4.42) approximately by using the straight line approximation to the trajectory (solution to (4.41) with $A = B = 1$)

$$r^2 = b_o^2 + (t_o - t)^2 \tag{4.43}$$

where b_o, t_o are constants. There are two cases.

Case A. If $\beta = \frac{t_o}{t}$, $\eta = -1$, then

$$\frac{1}{b}\ \frac{db}{dt} = \left(\frac{GM}{t}\right)_o\ \frac{1}{r} \tag{4.44}$$

integrates readily to

$$\frac{b}{b_o} = \left[\frac{t-t_o}{b_o} + \sqrt{1 + \left(\frac{t-t_o}{b_o} \right)^2} \,\right]^{\left(\frac{GM}{t} \right)_o} \tag{4.45}$$

Case B. If $\beta = \frac{t}{t_o}$, $\eta = 1$, then

$$\frac{1}{b} \frac{db}{dt} = - \frac{(GM\ t)_o}{r\ t^2} \tag{4.46}$$

gives

$$\log \frac{b}{b_o} = \left(\frac{GM}{t} \right)_o \left\{ \frac{b_o(t-t_o)}{t_o^2} - \log \frac{t_o}{t} \left(1 + \frac{t-t_o}{b_o} \right) \right\} \tag{4.47}$$

Since $\left(\frac{GM}{t} \right)_o$ is extremely small, being of the order

$$\frac{H_o}{c} \frac{(GM)_o}{c^2} = \frac{\text{gravitational radius of a star}}{\text{radius of universe}} \tag{4.48}$$

we can write in both cases

$$b \approx b_o \text{ , constant} \tag{4.49}$$

Equation (4.41) is then formally the same as in standard theory, and gives the same expressions for the time required to go from $r_0 (r_0 = b_0 B^{\frac{1}{2}}(r_0) \approx b_0)$ to r, i.e.,

$$t(r, r_0) = \sqrt{r^2 - r_0^2} + 2\,GM \log \frac{r - \sqrt{r^2 - r_0^2}}{r_0} + GM \sqrt{\frac{r - r_0}{r + r_0}} \qquad (4.50)$$

with the proviso that MG goes like β^{-1}.

The maximum round-trip time, when the planets are in superior conjunction and the ray just grazes the sun $(r_0 \approx R_s)$ is given by (M = Mercury, E = Earth, S = Sun)

$$\begin{aligned} \frac{1}{2}\, t_{max} = {} & \sqrt{r_E^2 - R_S^2} + \sqrt{r_M^2 - R_S^2} + 2\, M_S\, G \log \frac{r_E + \sqrt{r_E^2 - R_S^2}}{R_S} \\ & + 2\, M_S\, G \log \frac{r_M + \sqrt{r_M^2 - R_S^2}}{R_S} + M_S\, G \left(\frac{r_E - R_S}{r_E + R_S} \right)^{\frac{1}{2}} \qquad (4.51) \\ & + M_S\, G \left(\frac{r_M - R_S}{r_M + R_s} \right)^{\frac{1}{2}} \end{aligned}$$

where

r_E = radius of Earth's orbit

r_M = radius of Mercury's orbit

R_S = radius of Sun

In 4.2 we concluded that the semi-latus rectum and semi-major axis all vary like

$$r = r^0 \beta^{-1} \tag{4.52}$$

The round-trip time is then

$$t_{max} = 2\beta^{-1} \left\{ r_E^0 + r_M^0 + 2(M_S G)_o \left[1 + \log \frac{4 r_E^0 r_M^0}{R_S^2} - 2 \log \beta \right] \right\} \tag{4.53}$$

If we define the round-trip time as measured today by

$$t^0_{max} = 2 \left\{ r_E^0 + r_M^0 + 2(M_S G)_o \left[1 + \log \left(\frac{4 r_E^0 r_M^0}{R_S^{02}} \right) \right] \right\} \tag{4.54}$$

then

$$t_{max} = \beta^{-1} \left[t^0_{max} - 8(M_S G)_o \log \beta + 4(M_S G)_o \log \frac{R_S^0}{R_S} \right] \tag{4.55}$$

The maximum round-trip excess time delay is given by

$$(\Delta t)_{max} = t_{max} - 2\, r_E^0 - 2\, r_M^0$$

$$= \beta^{-1} (\Delta t)^0_{max} - 8\,\beta^{-1} (M_S\, G)_o \log \beta - 4\,\beta^{-1} (M_S\, G)_o \log \frac{R_S}{R_S^0} \tag{4.56}$$

$$+ (2\, r_E^0 + 2\, r_M^0)(\beta^{-1} - 1)$$

where

$$(\Delta t)^0_{max} = t^0_{max} - 2\, r_E^0 - 2\, r_M^0 \tag{4.57}$$

In equation (4.56) the second and fourth terms correspond to orbital expansion. The third term arises from changes in the solar radius and the first term is attributed to the change in GM.

Arguments based on homological transformations give the following relation for the radius of the sun

$$R \sim G^{g_1}\, M^{m_1} \tag{4.58}$$

where

$$g_1 = \frac{n + k_2 - 4}{n + 3 + 3\, k_1 + k_2}\,, \quad m_1 = \frac{n - 1 + k_1 + k_2}{n + 3 + 3\, k_1 + k_2}$$

and n, k_1 and k_2 are the indices in the nuclear source term and the opacity, i.e.

$$\epsilon = \epsilon_0 \, \rho \, T^n \, , \quad \tilde{k} = k_0 \, \rho^{k_1} \, T^{k_2}$$

For the p-p chain, n = 4.5, and for the case of Kramers' opacity, $k_1 = 1$ and $k_2 = -3.5$. With these values, we find

$$\log \frac{R_S}{R_S^0} = -\frac{3}{7} \log \frac{G}{G^0} + \frac{1}{7} \log \frac{M_S}{M_S^0} \qquad (4.59)$$

This value of $\log \frac{R_S}{R_S^0}$ may be used in equation (4.56).

4.4 Deflection of Light[29]. Conclusions

The photon trajectory is given by

$$\frac{1}{r^4} \left(\frac{dr}{d\varphi} \right)^2 = \frac{1}{b^2} - \frac{B}{r^2} \qquad (4.60)$$

We have seen in the previous section that during its entire journey, the "impact parameter" b is almost a constant, b_0.

We conclude that the deflection of a light ray is given by the standard expression $\frac{4\,GM}{b}$, except that since GM $\sim \beta^{-1}$, the deflection of light will increase or decrease with time depending on whether there is matter creation or not.

Pausing at this point to look backwards, we see that the first test, the perihelion precession of a planet per orbital period has a constant expression equal to the standard theory. But the orbital period is given by Kepler's law

$$
\begin{aligned}
T &= \frac{2\pi\, a^{3/2}}{(GM)^{\frac{1}{2}}} \\
&= T_0\, \beta^{-1} \left\{ 1 - \left[\frac{GM}{a(1-e^2)^{\frac{1}{2}}\, c^2} \right]_o \log \beta \right\}^{3/2}
\end{aligned}
\tag{4.61}
$$

where

$$
T_o = \frac{2\pi\, a_0^{3/2}}{(GM)_0^{\frac{1}{2}}} ,
$$

and so changes with time as β^{-1} . The net result is that the rate of perihelion advance of the planet is given by

$$
\Delta\dot{\omega} = (\Delta\dot{\varphi})_o\, \beta
$$

where

$$(\Delta\dot{\varphi})_0 = \frac{6\pi}{T_0}\left(\frac{GM}{L}\right)_0$$

For Mercury, this amounts to (43"/century) β.

The third test, the bending of light by the sun, suffers from the handicap that it cannot be monitored at all times.

The most promising test is the echo-delay experiment. Continuous measurements of this effect can be made for planets and artificial satellites near superior conjunction. Of course, hardly any measurement known to us has been so much distracted by the difficulty of distangling the relevant parameters from a maze of other celestial variables.

As our expressions have shown, the measured quantities at $t = t_0$ are the same as in the standard theory. Only by comparing data over a long interval of time, can a difference be established between the predictions of the gauge-covariant theory and those of the standard theory.

4.5 Planetary Orbits

Instrumentation technology has permitted a high accuracy measurement of planetary distances and orbital periods in atomic units. It has been suggested that such measurements could in the near future reveal deviations from predictions of the standard gravitational theory, such as the secular variation of the orbital period T of two gravitating bodies. In this section, we shall derive some predictions of the gauge covariant theory relevent to such measurements.

Denoting the orbital period and radius in Einstein units by $\bar{T}$, $\bar{R}$, from the transformation law (2.4), we have

$$\bar{T} = \beta T \ , \quad \bar{R} = \beta R \tag{4.62}$$

where T, R are the orbital period and radius in atomic units. Since $\bar{T}$ and $\bar{R}$ are constants, we find

$$\frac{\dot{T}}{T} = \frac{\dot{R}}{R} = - \frac{\dot{\beta}}{\beta} \tag{4.63}$$

We have already seen that the macroscopic mass M of an object satisfies the relation (see eqs. (2.55), (2.47a))

$$GM \sim \beta^{-\Pi(G)} \beta^{\Pi(G) - 1} = \beta^{-1}$$

Consequently,

$$\frac{\dot{T}}{T} = \frac{\dot{R}}{R} = \frac{(GM)^{\cdot}}{GM} \tag{4.64}$$

It should be noted that in the gauge covariant theory of gravitation, the product GM rather than the gravitational constant alone causes the variations of orbital periods and radii. If continuous creation exists, (2.56) and (4.63) give

$$\frac{\dot{T}}{T} = \frac{\dot{R}}{R} = \frac{1}{t} \tag{4.65}$$

The above equations immediately yield

$$T, R \sim t \qquad \text{(matter creation)} \tag{4.66}$$

On the other hand, if only the gravitational constant varies and no matter creation is postulated, (2.56a) and (4.63) yield

$$\frac{\dot{T}}{T} = \frac{\dot{R}}{R} = -\frac{1}{t} \tag{4.67}$$

(no matter creation)

$$T,\ R \sim t^{-1} \tag{4.67a}$$

Equation (4.63) from which the relations (4.65) or (4.67a) have been derived, can also be obtained by considering the equation of orbital motion. From (4.7) and (4.8), with $A = B = 1$ and $p = t$, we obtain

$$\ddot{r} - r\dot{\varphi}^2 + \frac{GM}{r^2} = -\dot{r}\,\frac{\dot{\beta}}{\beta} \tag{4.68a}$$

$$\ddot{\varphi} + \frac{2}{r}\,\dot{\varphi}\,\dot{r} = -\dot{\varphi}\,\frac{\dot{\beta}}{\beta} \tag{4.68b}$$

(4.68b) leads immediately to

$$J\beta = \beta r^2\,\dot{\varphi} \equiv h = \text{constant} \tag{4.69}$$

which if substituted into (4.68a) gives

$$\ddot{r} - \frac{h^2}{\beta^2\,r^3} + \frac{GM}{r^2} = -\dot{r}\,\frac{\dot{\beta}}{\beta} \tag{4.70}$$

We define a new variable $U \equiv \frac{1}{r}$ and use equation (4.69) again; (4.70) can be rewritten as

$$\frac{d^2 U}{d\varphi^2} + U = \frac{GM\,\beta^2}{h^2} \tag{4.71}$$

The above is recognized as the standard equation for an elliptical orbit if the right hand side is a constant. But since we expect the time scales of variations of G, M and β to be long compared to the orbital period, we can consider the orbit as describing an ellipse whose parameters undergo secular variations as dictated by the r.h.s. of (4.71). For simplicity, we consider only a circular orbit whose radius can be found from (4.71) to be

$$R = \frac{h^2}{GM\,\beta^2} \tag{4.72}$$

Elimination of h from (4.72) and (4.69) gives Kepler's third law

$$\left(\frac{2\pi}{T}\right)^2 R^3 = GM \;, \quad \left(\dot{\varphi} \equiv \Omega = \frac{2\pi}{T}\right) \tag{4.73}$$

in its standard form. Differentiation of (4.69) and (4.73) gives

$$2\,\frac{\dot{R}}{R} + \frac{\dot{\beta}}{\beta} - \frac{\dot{T}}{T} = 0 \tag{4.74}$$

$$3 \frac{\dot{R}}{R} - 2 \frac{\dot{T}}{T} - \frac{(GM)^{\cdot}}{GM} = 0 \tag{4.75}$$

It should be remarked that the second derivation while more elaborate is by no means logically more complete than the first. We have presented both derivations to emphasize consistancy of our reasoning at a transformation of units and caution against ad hoc introduction[15] of dynamical equations without a sound theoretical framework.

Let us rewrite equations (4.74) and (4.75) as follows:

$$\frac{\dot{T}}{T} = - 2 \frac{(GM)^{\cdot}}{GM} - 3 \frac{\dot{\beta}}{\beta} \tag{4.76}$$

$$\frac{\dot{R}}{R} = - \frac{(GM)^{\cdot}}{GM} - 2 \frac{\dot{\beta}}{\beta} \tag{4.77}$$

In the framework of the covariant theory, we found earlier that

$$\frac{\dot{\beta}}{\beta} = - \frac{(GM)^{\cdot}}{GM}$$

Hence,

(a) with matter creation, $M \sim t^2$,

$$\frac{\dot{T}}{T} = \frac{(GM)^{\cdot}}{GM} = \frac{1}{t} \tag{4.78a}$$

$$\frac{\dot{R}}{R} = \frac{(GM)^{\cdot}}{GM} = \frac{1}{t} \tag{4.78b}$$

(b) without matter creation

$$\frac{\dot{T}}{T} = \frac{\dot{G}}{G} = - \frac{1}{t} \tag{4.79a}$$

$$\frac{\dot{R}}{R} = \frac{\dot{G}}{G} = - \frac{1}{t} \tag{4.79b}$$

There exists in the literature what Dirac[10] has called a primitive theory of variable gravitational constant. In this theory, the dynamic equations of general relativity or their Newtonian limits are considered valid, but G is allowed to be a function of time. This theory also gives variations of T and R as a result of the variation of G. In fact, the results are obtainable as a limit of equations (4.74)-(4.75) when β is a constant. Thus we find for the primitive theory with M and β constant,

$$\text{(c)} \qquad \frac{\dot{T}}{T} = 2\frac{\dot{R}}{R} = -2\left(\frac{\dot{G}}{G}\right)_{pr} , \tag{4.80}$$

which are the relations used by Shapiro[32] and Van Flandern[33].

Comparison with (4.78)-(4.79) shows that for any given measurement of $\dot{T}/T$, the gauge covariant theory gives different interpretations for the variation of GM than the primitive theory. However, if $\dot{R}/R$ and $\dot{T}/T$ are measured simultaneously, (provided sufficient accuracy is attained), one can in principle distinguish the two theories, since the ratio of the two measurements should be 1 for the gauge covariant theory and 1/2 for the primitive theory.

Given the present age of the universe to be of the order of ten billion years, we expect from (4.78) and (4.79) that the measured fractional variation of the orbital period would be $\sim \pm 1 \times 10^{-10}$ year^{-1}, where the positive sign applies if multiplicative matter creation exists. Recently, Van Flandern[33] reported a measured value of

$$\frac{\dot{T}}{T} = (15 \pm 5.4) \times 10^{-11} \text{ year}^{-1} \tag{4.81}$$

for the difference between the atomic period and the ephemeris period of the moon. If verified, the gauge covariant theory would imply the existence of matter creation. However, confirmation of Van Flandern's result has not been forthcoming.

4.6 Stellar Structure Equations

In the gauge covariant theory of gravitation, one accepts the possibilities of a gradual weakening of the gravitational field and continuous matter creation. Thus, a star in hydrodynamic equilibrium may undergo secular variations induced by the variations of gravitational field strength and the total mass of the star. In this section, we apply the field equations (2.34) to the problem of stellar structure. Assuming spherical symmetry as usual, the line element and the velocity field can be written as

$$ds^2 = e^{2\phi(t,r)}\, dt^2 - e^{2\psi(t,r)}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{4.82}$$

$$u^\mu = (\gamma\, e^{-\phi}\ ,\ (\gamma^2 - 1)^{1/2}\ e^{-\psi}, 0, 0) \tag{4.83}$$

where γ is a function of r and t. It can be easily seen that

$$u^\mu\, u_\mu = g_{\mu\nu}\, u^\mu\, u^\nu = 1$$

After some lengthy algebra, the non-trivial field equations are, dropping terms involving the cosmological constant

$$\frac{\partial}{\partial r}\left[r(1-e^{-2\psi})\right] = 2G\{4\pi r^2(\rho\gamma^2 + p(\gamma^2 - 1))\}$$

$$+ \left(2\,\frac{\beta''}{\beta} + 4\,\frac{1}{r}\,\frac{\beta'}{\beta} - 2\psi'\,\frac{\beta'}{\beta} + \frac{\beta'^2}{\beta^2}\right) r^2\, e^{2\psi}$$

$$- \left(3\,\frac{\dot{\beta}^2}{\beta^2} + 2\,\frac{\dot{\beta}}{\beta}\,\dot{\psi}\right) r^2\, e^{-2\phi} \tag{4.84a}$$

$$\frac{e^{-\psi}\,\dot{\psi}}{r} = 4\pi G(\rho + p)\,\gamma(\gamma^2 - 1)^{1/2}\, e^{\phi} + \left(\frac{\dot{\beta}'}{\beta} - \frac{\dot{\beta}}{\beta}\,\phi' - \frac{\beta'}{\beta}\,\dot{\phi} - 2\,\frac{\dot{\beta}\,\beta'}{\beta^2}\right) e^{-\psi} \tag{4.84b}$$

$$\frac{2e^{-2\psi}\,\phi'}{r} - \frac{1-e^{-2\psi}}{r^2} = 8\pi G\left[p\gamma^2 + \rho(\gamma^2 - 1)\right] + \left(2\,\frac{\ddot{\beta}}{\beta} - 2\,\frac{\dot{\beta}}{\beta}\,\dot{\phi} - \frac{\dot{\beta}^2}{\beta^2}\right) e^{-2\phi}$$

$$- 2\left(\phi' + \frac{4}{r}\right)\frac{\beta'}{\beta}\, e^{-2\psi} \tag{4.84c}$$

$$\left[\ddot{\psi} + 2\,\frac{\ddot{\beta}}{\beta} + \dot{\psi}^2 - \dot{\psi}\,\dot{\phi} + \frac{\dot{\beta}}{\beta}\left(2\dot{\psi} - 2\dot{\phi} - \frac{\dot{\beta}}{\beta}\right)\right] e^{-2\phi}$$

$$= 8\pi G\, p + \left(\phi'' + \phi'^2 + \frac{\phi'}{r} - \frac{\psi'}{r} - \phi'\,\psi' + 2\,\frac{\beta'}{\beta}\,\phi' + \frac{2}{r}\,\frac{\beta'}{\beta} - 2\,\frac{\beta'}{\beta}\,\psi'\right.$$

$$\left. + 2\,\frac{\beta''}{\beta} - \frac{\beta'^2}{\beta^2}\right) e^{-2\psi} \tag{4.84d}$$

where dots and primes stand for partial differentiation with respect to t and r.

We have formally included the radial as well as time dependence of β. But we must point out here the main difficulty that remains ahead for the gauge covariant theory. In §2.6, we used the LNH for the determination of β in the context of homogeneous cosmology, and the question of spatial variation of β did not arise. More generally, there is no a priori reason to exclude a spatial variation, and the LNH is therefore insufficient for the determination of the gauge function. In this sense the theory is incomplete. However, if one considers the time variation of β indicated in §2.6 as a cosmological effect and if one is interested in the cosmological feed back to local physics, such as the problem of stellar structure considered here, it appears reasonable to use the cosmologically determined β for local computations. In the following we shall assume $\beta = \beta(t)$.

The field equations (4.84) can be further simplified if we make the slow motion approximation. Dropping v^2 terms, we have $\gamma \simeq 1$. Equations (4.84a), (4.84b) and (4.84c) become

$$\frac{\partial}{\partial r}\left[r(1-e^{-2\psi})\right] = 2G(4\pi r^2 \rho) - \left(3\frac{\dot\beta^2}{\beta^2} + 2\frac{\dot\beta}{\beta}\dot\psi\right) r^2 e^{-2\phi} \qquad (4.85a)$$

$$\dot\psi = -\frac{\dot\beta}{\beta} r\phi' \qquad (4.85b)$$

$$\frac{2\phi'}{r} e^{-2\psi} - \frac{1-e^{-2\psi}}{r^2} = 8\pi G p + \left(2\frac{\ddot\beta}{\beta} - 2\frac{\dot\beta}{\beta}\dot\phi - \frac{\dot\beta^2}{\beta^2}\right) e^{-2\phi}$$

(4.85c)

Using (4.85b), (4.85a) can be integrated to yield

$$e^{-2\psi} = 1 - 2 \frac{GM}{r} + r^2 e^{-2\phi} \left(\frac{\dot{\beta}}{\beta} \right)^2 \tag{4.86}$$

where

$$M(t,r) = 4\pi \int_0^r \rho(t,r') r'^2 \, dr' \tag{4.87}$$

Since we are considering cosmologically induced variations of stellar structure, $\dot{\beta}/\beta$ is of order $1/\tau_o$, where τ_o is the age of the universe. For a local system, $(r/t_o)^2 << 1$ (we have put the velocity of light $c = 1$), so

$$e^{-2\psi} \simeq 1 - 2 \frac{GM}{r} \tag{4.86a}$$

which is formally analogous to the standard stellar equilibrium solution in general relativity. But in the present context both G and M are functions of time.

In the same approximation, (4.85c) and the radial component of (2.48) can be written as

$$\begin{aligned} \phi' &\cong \frac{G}{r} \left(\frac{M + 4\pi r^3 p}{r - 2GM} \right) \\ \frac{dp}{dr} &\cong - \phi'(\rho + p) \end{aligned} \tag{4.88}$$

where (4.86a) has been used. We finally arrive at the stellar structure equation

$$\frac{dp}{dr} \cong - \frac{G}{r} \frac{(\rho + p)(M + 4\pi r^3 p)}{(r - 2GM)} \tag{4.89}$$

This equation indicates again that any cosmologically induced variation of stellar structure is, to an accuracy of (r/t_o), implicitly contained in the variation of G and M. Consequently, classical results such as the luminosity of a star[12, 13]

$$L \sim G^7 M^5 \mu^4 \tag{4.90}$$

and the polytrope relation[12, 13] ($p \sim \rho^\Gamma$, $\Gamma = 1 + 1/n$, n = polytropic index),

$$R^{3\Gamma-4} GM^{2-\Gamma} = \text{const.} \ , \tag{4.91}$$

remain valid up to the same accuracy.

4.7 Surface Temperature of the Earth - Geological Effects

Cosmological ideas such as the ones presented in this paper are sometimes tested using arguments based on the acceptable temperature of the earth in the past few billion years. Several arguments against and in favour of a time varying G have been published over the years but no firm conclusion can yet be reached.

The absolute luminosity of the sun is known to vary as[12,13]

$$L \sim G^7 M^5 \mu^4 \tag{4.92}$$

where μ is the mean molecular weight. Defining an effective temperature as

$$\sigma T_{eff}^4 = \Phi = \frac{L}{4\pi R^2} \tag{4.93}$$

where R is the Sun-Earth distance, it is easy to see that even if G and M are constant, the temperature was lower in the past since μ was smaller. Simple computation show that 3×10^9 years ago, $T_{eff} = 230°K$, much lower than what has been indicated by geological data[34], even if one adds to T_{eff} about 30°K due to CO_2-H_2O green house effect. If $G \sim t^{-1}$ and M remain constant, the smaller μ in the past is more than balanced by the variation of G and one can get $T_{eff} = 360°K$ at 3×10^9 years ago. If however M also varies, $M \sim t^2$, the situation is again similar to the original case, with lower L and T_{eff} in the past.

We should point out however that the above estimates of T_{eff}, evaluated from (4.93) cannot be directly compared with temperatures derived from geological data since the green house effect and possibly other geo-thermal effects have been ignored, so that T_{eff} does not represent the physical temperature at the surface of the earth. Adjusting the chemical composition of the atmosphere by introducing a small amount of ammonia, Sagan and Mullen[35] were able to get such a large green house effect that the lower luminosity in the past was amply compensated for, and a higher "surface" temperature obtained. Consequently, the past thermal history of the earth cannot be used to argue conclusively for or against Dirac cosmology by estimating the variation of the solar constant alone. A more thorough analysis of the problem, including the varying green house effect with a varying solar constant, is now being attempted and the results will be published elsewhere.

Other geophysical effects of a varying G cosmology are often discussed and we shall limit our discussion here to showing that the present theory does not contradict any well-established fact.

An update survey of implications for geophysics as arising from non-standard cosmologies can be found in a paper by Wesson[36]. We shall discuss here two major effects: the expansion of the Earth radius and the spin-down. Thorough discussion and pertinent references can be found in the paper by Wesson.

Having shown that in the present theory the hydrodynamic equations governing the stability of a star are unaffected by the gauge-function $\beta(t)$, we can write down the expression to be satisfied by R, G and M, Eq. (4.91)

$$R \sim M^{\frac{\Gamma-2}{3\Gamma-4}} \quad G^{-\frac{1}{3\Gamma-4}} \tag{4.94}$$

which yields the desired results for the time variation of the Earth's radius, namely

A) Matter Creation

$$\dot{R}(t_o) = \frac{2\Gamma-3}{3\Gamma-4} \frac{R(t_o)}{t_o} = \frac{2\Gamma-3}{3\Gamma-4} \left\{ \begin{array}{l} .425 \\ .354 \\ .319 \end{array} \right. \left(\frac{mm}{yr}\right) \tag{4.95}$$

B) No Matter Creation

$$\dot{R}(t_o) = \frac{1}{3\Gamma-4} \frac{R(t_o)}{t_o} = \frac{1}{3\Gamma-4} \left\{ \text{Same} \right. \tag{4.96}$$

for t_o = 15, 18 and 20 billion years. Several independent determinations (Wesson discusses 21 of them) lead to the result

$$\dot{R}(t_o) = (.5 - .6) \text{ mm/yr} \tag{4.97}$$

for the last 500 million years.

Let us now look at the spin-down effect. It seems to be an accepted fact that the Earth is not only expanding but also slowing down at a rate of

$$1.6 \ \frac{\text{msec}}{\text{century}} \tag{4.98}$$

We have already proven that within the gauge-covariant theory, the conserved angular momentum is given by Eq. (4.69), from which we deduce that

$$P \sim \beta R^2$$

or

$$\frac{\dot{P}}{P} = \frac{\dot{\beta}}{\beta} + 2 \frac{\dot{R}}{R}$$

or finally using Eqs. (4.95) and (4.96)

A)
$$\dot{P} = \frac{\Gamma-2}{3\Gamma-4} \frac{P}{t_o} = \frac{\Gamma-2}{3\Gamma-4} \left\{ \begin{array}{l} .576 \\ .480 \\ .432 \end{array} \right. \frac{\text{msec}}{\text{century}} \tag{4.99}$$

B)
$$\dot{P} = \frac{3\Gamma-3}{3\Gamma-4} \frac{P}{t_o} = \frac{3\Gamma-3}{3\Gamma-4} \left\{ \begin{array}{l} .576 \\ .480 \\ .432 \end{array} \right. \frac{\text{msec}}{\text{century}} \tag{4.100}$$

again for t_o = 15, 18 and 20 billion years.

Since the Earth is slowing down, $\dot{P}$ must be positive, which implies that

A) $\Gamma > 2$ or $\Gamma < 4/3$

B) $\Gamma > 4/3$ or $\Gamma < 4/3$

By fitting an expression of the type $p = a\rho^{\Gamma}$ to the numerical values of p and ρ for the Earth[37], one concludes that

$$4.5 \leq \Gamma \leq 7$$

The coefficient $(2\Gamma-3)/(3\Gamma-4)$ changes very little for $\Gamma = 5$ or $\Gamma = 7$. In one case is .636, in the other .647. We shall take .64, so that

A) $$\dot{R}(t_o) = \begin{cases} .272 \\ .226 \\ .204 \end{cases} \quad \text{(mm/yr)} \tag{4.101}$$

For case B), $1/(3\Gamma-4)$ varies from 1/11 to 1/17 for $\Gamma = 5$ and $\Gamma = 7$, so that

B) $$\dot{R}(t_o) = \begin{array}{cc} (\Gamma = 5) & (\Gamma = 7) \\ \begin{cases} .038 \\ .032 \\ .029 \end{cases} \text{(mm/yr)} & \begin{cases} .025 \\ .021 \\ .019 \end{cases} \text{(mm/yr)} \end{array} \tag{4.102}$$

In case A), cosmology contributes from 40% to 50% to the total value (4.97) depending on the age of the Universe. In case B) the contribution is only of few percents

The variation of $\dot{P}$ with Γ is even smaller. In fact, for case A) the coefficient $(\Gamma-2)/(3\Gamma-4)$ varies between .272 and .294. For case B), $(3\Gamma-3)/(3\Gamma-4)$ varies between 1.091 and 1.059. We shall take .28 for the first case and 1.07 for the second. We then have

$$\text{A)} \qquad \dot{P}(t_o) = \left\{ \begin{array}{l} .161 \\ .134 \\ .121 \end{array} \right. \left(\frac{\text{msec}}{\text{century}}\right) \tag{4.103}$$

$$\text{B)} \qquad \dot{P}(t_o) = \left\{ \begin{array}{l} .616 \\ .514 \\ .462 \end{array} \right. \left(\frac{\text{msec}}{\text{century}}\right) \tag{4.104}$$

Case A) contributes at most 10%, whereas case B) can contribute as much 38%. Either case is found to lead to perfectly admissible results.

V. FINAL REMARKS

In this paper we have presented a gauge covariant theory of gravitation, characterized by a set of equations which are complete only after a choice is made of the gauge function $\beta(t)$. Among an a priori infinite number of choices, two seem particularly appropriate: Einstein gauge (β = const) and atomic gauge.

Since no general principle has yet been given as to how to choose $\beta(t)$ in atomic units, we have suggested the use of the large dimensionless numbers relating atomic and gravitational constants. Several results, ranging from cosmology, planetary orbits, stellar structure and earth's geology are then derived and shown to be consistent with a variety of well-known facts.

Even though such proofs of consistency must be given, they constitute a necessary but not sufficient raison d'etre for such a new theory.

Other more fundamental reasons exist which justify the study of a covariant theory of gravitation. The generalization is being pursued having in mind the relation between gravitational and atomic phenomena, a relation that in spite of having been discussed in the scientific literature with increasing frequency has not yet led to a satisfactory picture. Gravity is however being considered in a much broader light and its hoped-for relation to the structure of matter is more closely investigated, the ultimate goal being the unification of all types of interactions, and endeavor that has been recently crowned by encouraging success.

From the theoretical point of view, Weinberg and Salam have convincingly conjectured that electromagnetism and weak interactions can be combined into a unique non-Abelian gauge theory. Experimental evidence is so far in favor of such a theory. (Einstein theory of gravity is also non-Abelian). From the experimental point of view, strong interactions have recently been shown to exhibit scale invariance, a property so far possessed only by electromagnetic interactions.

Seemingly dividing barriers have either fallen or become more brittle upon close inspection and the gate seems to have finally opened to a flood of new interesting though still unrelated proposals.

In this paper we have focused our attention on a direction so far unexplored, namely scale invariance. We do not claim to have shown that gravity must be scale invariant, but only that a gravitational theory endowed with such a property, leads to no contradictions with well established facts ranging from geology to cosmology.

Since local gravitational phenomena have been historically the major cause of the high rate of casualties for other generalizations of Einstein equations, we have given a detailed presentation of the three classical tests, with the result that at any given instant of time the present theory yields the same results as ordinary standard theory.

Having passed that hurdle, we have indicated how the present theory can enlarge our interpretation of several phenomena, not ultimately being the only consistent theoretical framework which can accommodate a possible variation of the gravitational constant with cosmological time, a possibility entirely excluded by ordinary Einstein equations.

Besides passing several crucial tests, a theory must also be able to make predictions. In this respect we believe that the present theory can solve what has been a major difficulty concerning the cosmological constant Λ, within the framework of gauge-fields and broken symmetries. Although it is not known whether Λ is needed to explain cosmological facts like the magnitude vs. red-shift relations, it is unquestionably true that the stability of galactic clusters put limits on its magnitude. In fact Λ must be less than 10^{-57} cm^{-2}.

Since the cosmological constant Λ can physically be interpreted as the vacuum contribution to the energy momentum tensor of matter[38], it is possible to derive the following expression[39,40,41] within the framework of the gauge-fields,

$$\Lambda = -\frac{\pi}{\sqrt{2}} \frac{Gm_\varphi^2}{G_F} < 10^{-6} \text{ cm}^{-2}$$

some 50 orders of magnitude larger than the previous value. This large discrepancy, which has even been considered as undermining the credibility of the Higgs mechanism,[39] can be drastically reduced if not totally accounted for in the present theory. In fact, on the basis of (2.22) and (2.56), Λ must have a time dependence of the form

$$\Lambda(t) = \Lambda_o \left(\frac{t_o}{t}\right)^2$$

If $\Lambda_o < 10^{-57}$ today, $\Lambda(t) < 10^{-6}$ was achieved at $t \approx 10^{-8}$ sec, a time not drastically different from the quoted $t \approx 10^{-14}$ sec, i.e. $T \approx 300$ Gev at which the computations is usually performed. The computations can be improved further once we have a better understanding of the behavior of $\beta(t)$ at early cosmological times. In fact we have reasons to believe that $\beta(t)$ scales faster than t^{-1}, thus moving 10^{-8} sec to earlier times.

The early time dependence of $\beta(t)$ can be searched for by studying, for instance, nucleosynthesis ($t \approx 100$ sec) and demanding agreement with the observed abundance of He^4 and D. Conversely, one could look for a $\beta(t)$ at early times that, together with (2.56), accounts exactly for the 50 orders of magnitudes, if one has reasons to believe that the gauge-fields physics is indeed the relevant physical scenario governing early cosmological times.

An alternative is to fix $\beta(t)$ by trying to avoid the Big Bang singularity. Two of the authors have been working in that direction and preliminary results are encouraging. A $\beta(t)$ can indeed be found that satisfies the condition $\ddot{R}(0) > 0$, with $3\rho + p > 0$. The whole question concerning the role of the present theory in the early cosmological times remains to be studied.

In conclusion, a great deal of future work remains to be done both from the standpoint of internal consistency, comparison and relation with theories of fundamental interactions as well as direct comparison with observations.

It is our feeling however that the preceding analysis has shown how a gauge covariant theory can enlarge the possibilities of making one step further towards a unified theory of the various kinds of interactions, without contradicting any well accepted facts.

Appendix I

Co-Tensor Analysis

In this section, we shall first review the essential features of Weyl's geometry. Co-tensors are then defined in Weyl space. Some mathematical relations in co-tensor analysis, pertinent to the main text of this paper will be derived here.

The fundamental postulates of Weyl geometry are:

(A) There exist affine connections ${}^{*}\Gamma^{\mu}_{\nu\lambda}$ such that parallel transport of a vector ξ^{μ} can be defined as:

$$d\xi^{\mu} = - {}^{*}\Gamma^{\mu}_{\nu\lambda} \xi^{\nu} dx^{\lambda} \tag{A1.1}$$

where

$${}^{*}\Gamma^{\mu}_{\nu\lambda} = {}^{*}\Gamma^{\mu}_{\lambda\nu} \tag{A1.2}$$

(B) The change of length of a vector by parallel transport is given by

$$d(g_{\mu\nu} \xi^{\mu} \xi^{\nu}) = 2 g_{\mu\nu} \xi^{\mu} \xi^{\nu} k_{\lambda} dx^{\lambda} \tag{A1.3}$$

Note that the metrical properties of Weyl space are specified by both $g_{\mu\nu}$ and k_λ. Since lengths are not assummed to be preserved, the scale vector k_λ gives their variation under parallel transport.

Let

$$^*\Gamma_{\mu,\,\nu\lambda} = g_{\mu\rho}\ {}^*\Gamma^{\rho}_{\nu\lambda} \qquad \text{(A1.4)}$$

It can be easily shown that the affine connections are related to the metric and scale tensors by

$$^*\Gamma_{\mu,\nu\lambda} = \tfrac{1}{2}\,(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}) - (g_{\mu\nu}\,k_\lambda + g_{\mu\lambda}\,k_\nu - g_{\nu\lambda}\,k_\mu) \qquad \text{(A1.5a)}$$

Hence

$$^*\Gamma^{\mu}_{\nu\lambda} = \tfrac{1}{2}\,g^{\mu\rho}\,(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}) - (g^{\mu}_{\nu}\,k_\lambda + g^{\mu}_{\lambda}\,k_\nu - g_{\nu\lambda}\,k^{\mu})$$

$$= \Gamma^{\mu}_{\nu\lambda} - (g^{\mu}_{\nu}\,k_\lambda + g^{\mu}_{\lambda}\,k_\nu - g_{\nu\lambda}\,k^{\mu}) \qquad \text{(A1.5)}$$

where $\Gamma^{\mu}_{\nu\lambda}$ are the Christoffel symbols defined in terms of $g_{\mu\nu}$ as in Riemannian geometry. As usual $f_{,\lambda} = \partial f/\partial x^{\lambda}$.

If we define a curvature tensor in Weyl space by means of parallel displacement of a vector along a closed curve, we get analogous to the Riemannian case

$$ {}^{*}R^{\mu}_{\nu\lambda\rho} = \frac{\partial\, {}^{*}\Gamma^{\mu}_{\nu\lambda}}{\partial x^{\rho}} - \frac{\partial\, {}^{*}\Gamma^{\mu}_{\nu\rho}}{\partial x^{\lambda}} + {}^{*}\Gamma^{\eta}_{\nu\lambda}\, {}^{*}\Gamma^{\mu}_{\eta\rho} - {}^{*}\Gamma^{\eta}_{\nu\rho}\, {}^{*}\Gamma^{\mu}_{\lambda\eta} \tag{A1.6} $$

The associated contracted tensors ${}^{*}R_{\mu\nu}$ and ${}^{*}R$ can be written as

$$ \begin{aligned} {}^{*}R_{\mu\nu} &= {}^{*}R^{\lambda}_{\mu\lambda\nu} \\ &= R_{\mu\nu} - 2(k_{\mu;\nu} - k_{\nu;\mu}) - (k_{\mu;\nu} + k_{\nu;\mu}) \\ &\quad - g_{\mu\nu}\, k^{\lambda}_{;\lambda} - 2\, k_{\mu}\, k_{\nu} + 2\, g_{\mu\nu}\, k^{\lambda}\, k_{\lambda} \end{aligned} \tag{A1.7} $$

$$ {}^{*}R = g^{\mu\nu}\, {}^{*}R_{\mu\nu} = R - 6\, k^{\lambda}_{;\lambda} + 6\, k^{\lambda}\, k_{\lambda} \tag{A1.8} $$

where $R_{\mu\nu}$, and R are the Ricci tensor and scalar curvature defined in terms of $g_{\mu\nu}$ Clearly, if $k_{\mu} = 0$, the affine connections as well as the curvature tensors reduce to the Riemannian case and Weyl space in this limit becomes Riemannian space. We note also that ";" is used in this paper to denote the normal covariant differentiation, defined using $\Gamma^{\mu}_{\nu\lambda}$ rather than ${}^{*}\Gamma^{\mu}_{\nu\lambda}$.

Next consider a general scale transformation of the form

$$ ds \rightarrow ds' = \ell(x)\, ds \tag{A1.9} $$

Since

$$ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu \tag{A1.10}$$

and dx^μ being a coordinate differential does not change under scaling, we have

$$g_{\mu\nu} \rightarrow g'_{\mu\nu} = \ell^2(x) \, g_{\mu\nu} \tag{A1.11}$$

(A1.11) can be recognized as a conformal transformation. We remark that given (A1.10) as the definition of the line element, conformal transformation and scale transformation imply each other. The latter is also called a gauge transformation and we shall be using these terminologies interchangibly in this paper.

From (A1.3) it can be shown that under the scale transformation (A1.9), k_μ transform as follows:

$$k_\mu \rightarrow k'_\mu = k_\mu + (\ln \ell)_{,\mu} \tag{A1.12}$$

It is easy to show using (A1.11) and (A1.12) that ${}^*\Gamma^\mu_{\nu\lambda}$ is invariant under gauge transformation. It is of course not a tensor. But the tensor properties of ${}^*R^\mu_{\nu\lambda\rho}$, ${}^*R_{\mu\nu}$, ${}^*R$ can be easily established. Furthermore, since ${}^*\Gamma^\mu_{\nu\lambda}$ is gauge invariant, inspection of (A1.6) and (A1.7) shows that ${}^*R^\mu_{\nu\lambda\rho}$ and ${}^*R_{\mu\nu}$ are also gauge invariant.

Now we introduce the notion of a co-tensor. Let A denote a tensor of arbitrary rank, i.e. is under coordinate transformations, A has tensor properties. If in addition, under gauge transformation (A1.9),

$$A \rightarrow A' = \ell^{\Pi} A \tag{A1.13}$$

then A is called a co-tensor of power Π. In particular, if $\Pi = o$, A is called an in-tensor. Thus, we see that $^{*}R^{\mu}_{\nu\lambda\rho}$, $^{*}R_{\mu\nu}$ are in-tensors. From (A.11), $g_{\mu\nu}$ is a co-tensor of power 2. Since $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, it is a co-tensor of power -2.

Clearly, products of co-tensors are again co-tensors. In particular, let A_1, A_2 be co-tensors of powers Π_1 and Π_2, thus

$$A = A_1 A_2$$

is a co-tensor of power $\Pi = \Pi_1 + \Pi_2$. Consequently, $^{*}R$ is a co-scalar of power -2. (In the present terminology, scalar and vector are special cases of tensors). We mention the obvious fact that not all tensors are co-tensors. For example, $R_{\mu\nu}$ and R do not transform like (A1.13) although they have tensor properties under coordinate transformations.

The extension of the concept of tensor to that of co-tensor requires a corresponding extension of covariant differentiation. It is clear that the covariant derivative of a co-tensor is in general not a co-tensor. Let S, V, T be co-tensors of power Π having ranks 0, 1 and 2 respectively. We define the co-covariant differentiation of these objects as follows:

$$S_{*\mu} \equiv S_{,\mu} - \Pi\, k_\mu\, S \tag{A1.14a}$$

$$V^{\mu}_{*\nu} \equiv V^{\mu}_{,\nu} + {}^{*}\Gamma^{\mu}_{\nu\lambda}\, V^{\lambda} - \Pi\, k_\nu\, V^{\mu} \tag{A1.14b}$$

$$V_{\mu*\nu} \equiv V_{\mu,\nu} - {}^{*}\Gamma^{\lambda}_{\mu\nu}\, V_{\lambda} - \Pi\, k_\nu\, V_{\mu} \tag{A1.14c}$$

$$A^{\mu\nu}_{*\lambda} \equiv A^{\mu\nu}_{,\lambda} + {}^{*}\Gamma^{\mu}_{\lambda\rho}\, A^{\rho\nu} + {}^{*}\Gamma^{\nu}_{\lambda\rho}\, A^{\mu\rho} - \Pi\, k_\lambda\, A^{\mu\nu} \tag{A1.14d}$$

$$A_{\mu\nu*\lambda} \equiv A_{\mu\nu,\lambda} - {}^{*}\Gamma^{\rho}_{\mu\lambda}\, A_{\rho\nu} - {}^{*}\Gamma^{\rho}_{\nu\lambda}\, A_{\mu\rho} - \Pi\, k_\lambda\, A_{\mu\nu} \tag{A1.14e}$$

Generalization to higher rank co-tensors is immediate. It can be easily seen from expressions (A1.14) that the co-covariant derivative of a co-tensor of power Π is again a co-tensor of the same power.

The following relations will be found useful:

$$V^{\mu}_{*\nu} = V^{\mu}_{;\nu} - (\Pi+1)\, k_\nu\, V^{\mu} + k^{\mu}\, V_{\nu} - g^{\mu}_{\nu}\, k_\lambda\, V^{\lambda} \tag{A1.15}$$

$$V^{\mu}_{*\mu} = V^{\mu}_{;\mu} - (\Pi+4)\, k_\mu\, V^{\mu} \tag{A1.16}$$

$$A^{\mu\nu}_{*\lambda} = A^{\mu\nu}_{;\lambda} - (\Pi+2)\, k_\lambda\, A^{\mu\nu} - g^{\mu}_{\lambda}\, k_\rho\, A^{\rho\nu} - g^{\nu}_{\lambda}\, k_\rho\, A^{\mu\rho} + k^{\mu}\, A^{\nu}_{\lambda} + k^{\nu}\, A^{\mu}_{\lambda} \tag{A1.17}$$

$$A^{\mu\nu}_{*\nu} = A^{\mu}_{;\nu} - (\Pi+5)\, k_{\nu}\, A^{\mu\nu} - k_{\rho}\, A^{\rho\mu} + k^{\mu}\, A^{\nu}_{\nu} \qquad \text{(A1.18)}$$

If $A^{\mu\nu} = A^{\nu\mu}$, we have

$$A^{\mu\nu}_{*\nu} = A^{\mu\nu}_{;\nu} - (\Pi+6)\, k_{\nu}\, A^{\mu\nu} + k^{\mu}\, A^{\nu}_{\nu} \qquad \text{(A1.19)}$$

The metric tensor $g_{\mu\nu}$ satisfies the relations

$$g_{\mu\nu*\lambda} = 0$$

$$g^{\mu\nu}_{*\lambda} = 0$$

The analogue of the Einstein tensor $G_{\mu\nu}$ is

$$^{*}G_{\mu\nu} = {}^{*}R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}\, {}^{*}R$$

Appendix II

Dirac's Large Number Hypothesis

In this appendix, we briefly outline Dirac's large number hypothesis (LNH) and some of its consequences. More detailed discussions can be found in Dirac's papers[6], [8], [10], [11]. A comprehensive summary has also been given by Canuto and Lodenquai[12].

The motivation of Dirac's hypothesis has been the coincidences among certain large dimensionless numbers first noted by Eddington[28], and has been known as the Eddington numbers. One of these is the ratio of electrostatic and gravitational forces between a proton and an electron.

$$N_1 = \frac{e^2}{G\, m_e\, m_p} = 2 \times 10^{39} \tag{A2.1}$$

A second number arises when the age of the universe, approximated by the reciprocal of the Hubble expansion parameter is divided by an atomic unit of time.

$$N_2 = \frac{m_e\, c^3}{H_o\, e^2} = 7 \times 10^{40} \tag{A2.2}$$

If the present average density of matter in the universe is taken to be $\rho = 10^{-30}$ gm cm^{-3}, the total mass within the visible universe defined by this Hubble radius c/H_o is given by $\frac{4\pi}{3}\, \rho \left(\frac{c}{H_o} \right)^3$. A third large number can thus be derived:

$$N_3 = \frac{4\pi}{3} \frac{\rho}{m_p} \left(\frac{c}{H_o}\right)^3 \simeq 10^{78} \qquad \text{(A2.3)}$$

The coincidences mentioned above refer to the fact that the following relations hold:

$$N_1 = a_1 N_2 \qquad \text{(A2.4a)}$$

$$N_3^{1/2} = a_3^{1/2} N_2 \qquad \text{(A2.4b)}$$

with a_1, a_3 being of order close to unity. Many theorists believe that the dimensionless constants in physics, such as $e^2/\hbar c$ or m_p/m_e can in principle be explained theoretically. Likewise there have been numerous speculations about the coincidences of the Eddington numbers. Dirac pointed out that the ratios N_1: N_2: $N_3^{1/2}$ are of order to unity. They are expected to be derivable theoretically as one would expect for the fine structure constant. Accepting this point of view, and noting that N_2 corresponds to the cosmological epoch, he came to the conclusion that the gravitational constant measured in atomic units, and the number of baryons in the visible universe must be a function of the epoch. Furthermore, he formulated the hypothesis that given any large dimensionless number N, it can be expressed as

$$N = a\, N_2^k \qquad \text{(A2.5)}$$

where a and k are constants of order unity. Clearly, equations (A2.4) are special cases of (A2.5). It should be noted that (A2.5) is now taken to be a functional relation: as time passes, N_2 necessarily changes and N would change accordingly.

The immediate consequence of the large number hypothesis is that the gravitational constant is inversely proportional to the epoch and the number of baryons in this visible universe increases like the square of the epoch. When Dirac[11] applied the LNH to R, the radius of the universe measured in atomic units, he concluded that the exponent k in (A2.5) must be 1 and hence

$$R \sim t \tag{A2.6}$$

where we have written t for N_2, which is the epoch in atomic units.

It should be emphasized that the large numbers considered thus far have been derived as ratios of macroscopic, gravitational units and microscopic, atomic units. In fact, this prompted Dirac[6] in his original article on the subject to suggest that the proper way to understand the LNH is by the consideration of two metrics. But this line of reasoning has not been taken up until recently.

Other astrophysical consequences of the LNH have been considered by various authors. The conclusions do not follow as simply from the LNH as do equations (2.4) and (2.6), and various dynamical relations had to be used implicitly or explicitly. Hence instead of summarizing these results here, we shall consider them anew in the main text as consequences of the modified dynamics of the gauge covariant theory of gravitation.

Acknowledgements

P. J. Adams, S.-M. Hsieh and E. Tsiang would like to thank Dr. R. Jastrow for his hospitality at the Institute for Space Studies. V. Canuto would like to thank Professor P. A. M. Dirac for suggestions and criticism.

References

(1) Weyl, H., Space, Time and Matter. Springer 1921.

(2) Pauli, W., Theory of Relativity. Pergamon Press, 1958.

(3) Eddington, A. S., Mathematical Theory of Relativity, Cambridge University Press, 1924.

(4) Callan, C. G., Coleman, S., Jackiw, R., Ann. Phys. 59, 42, 1970.

(5) Fulton, T., Rohrlich, F., Witten, L., Review of Mod. Phys. 34, 442, 1962.

(6) Dirac, P.A.M., Nature, 139, 323 1937.

(7) Dirac, P.A.M., Proc. Roy. Soc., London, A333, 403 1973.

(8) Dirac, P.A.M., Proc. Roy. Soc., London, A338, 439 1974.

(9) Dirac, P.A.M., Proc. Roy. Soc., A165, 199, 1938.

(10) Dirac, P.A.M., In Theories and Experiments in High Energy Physics, p. 443. Edited by B. Kursunoglu, A. Perlmutter and S. M. Widmayer. Plenum, 1975.

(11) Dirac, P.A.M., in Physicist's Conception of Nature. Ed. J. Mehra, Reidel, 1973.

(12) Canuto, V. and Lodenquai, J., Ap. J. to be published.

(13) Canuto, V., Submitted to Ap. J.

(14) Steigman, G., Nature 261, 479, 1976; For a rebuttal see Ref. (30).

(15) Roxburg, I. W., Nature 261, 301, 1976; For a rebuttal see Ref. (31)

(16) Eisenhart, L. P., Riemannian geometry. Princeton University Press 1926, pp. 89-91.

(17) Einstein, A., S. B. Preuss. Akad. Wiss., 478, 1918.

(18) See for example Pauli in reference (2), pages 195-196.

(19) See e.g. T. D. Lee, Cargese lectures in physics, Vol. 1, p. 55 for a discussion of invariance principles and non-measurability.

(20) Papapetrou, A. 1974, Lectures on general relativity, D. Reidel Publ. Co. Boston, Mass, pages 24-28.

(21) Bicknell, G. V., J. Phys. A, 9, 1077, 1976.

(22) Brans, C., Dicke, R. H., Phys. Rev. 124, 925, 1961; Dicke, R. H., 1962, Science 138, 653.

(23) Wagoner, R. V., Phys. Rev. D, 1, 3209, 1970.

(24) It should be noted that the relative signs for the terms in the equations of this section may be different from what appears in the papers cited. This is due to the fact that the sign convention we have chosen for $g_{\mu\nu}$ and $R^{\lambda}_{\mu\nu}$ is different.

(25) Deser, S., Ann. Phys., (N. Y.), 59, 248, 1970.

(26) Anderson, J. L., Phys. Rev. D, 3, 1689, 1971.

(27) Pietenpol, J. L., Incotal, R., Speiser, D., Phys. Rev. Lett., 33, 387, 1974.

(28) Eddington, A. S. The Philosophy of Physical Science, Cambridge, 1939.

(29) Weinberg, S., Gravitation and Cosmology. John Wiley and Sons, 1972.

(30) Canuto, V., Adams, P. J., Hsieh, S. H., Tsiang, E., Nature, in print, (1976).

(31) Canuto, V., Adams, P. J., Hsieh, S. H., Tsiang, E., Nature, in print, (1976).

(32) Shapiro, I. I., Science, 162, 352, 1968.

(33) Van Flandern, T. C., in Seventh Texas Symposium on Relativistic Astrophysics, New York Academy of Sciences, 1975.

(34) Knauth, L. P. and Epstein, S. 1976, Geochim. et Cosmochim. Acta 40, 1095.

(35) Sagan, C. and Mullen, G. 1972, Science, 777, 52.

(36) Wesson, P. Q. Jl. R. Astr. Soc. 14, 9, 1973.

(37) Stacey, F. D., Physics of the Earth, John Wiley & Sons, Inc. New York, 1969, Table E1, pp. 281-282.

(38) Zeldovich, Ya. B. 1968, Sov. Phys. Uspekhi, 11, 381.

(39) Linde, A. D. 1974, JETP Lett. 19, 183

Dreitlein, J. 1974, Phys. Rev. Lett. 33, 1243

Veltman, M. 1975, Phys. Rev. Lett. 34, 777

Bludman, S. and Ruderman, M. 1976, Columbia Preprint

(40) Kohler, D., Watson, B. A. and Becker, J. A. 1974, Phys. Rev. Lett. 33, 1628.

These authors show that m_φ cannot be in the range (1.03-18.2) Mev

(41) Sato, K. and Sato, H. 1975, Progr. Theor. Phys. 54, 1564.

These authors show that $m_\varphi < .7\ m_e$ is ruled out, otherwise the emission of Higgs mesons would affect the evolution of stars drastically.